ANALYTICAL AND NUMERICAL-BASED APPROACH TO PREDICT THE INFLUENCE OF MORPHOLOGICAL FLUCTUATIONS ON THE EFFECTIVE TRANSVERSE ELASTIC BEHAVIOUR OF A TRANSVERSELY RANDOM UD COMPOSITE

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Keywords: Micromechanical models, Morphological Reprepresentative Patterns, Generalized Self-Consistent Schemes, Fibre reinforced composites, Fluctuations of morphology

Abstract

This paper presents an enhanced micromechanical analytical model to predict the transverse elastic properties of unidirectionnal (UD) composite materials. In contrast to the elastic longitudinal constants, the transverse effective properties are quite dependent on the space fibre distribution. This model allows thus to take into account the effect of local morphological fluctuations, mainly induced by random fibre arrangements, on the effective transverse properties. For that purpose, a "*n*-phase" Generalized Self Consistent Scheme (GSCS) has been extended with a Morphologically Representative Pattern (MRP) approach. Analytical solutions based on two morphological parameters are provided to predict the *transverse shear modulus* and the *transverse bulk modulus* of any UD composite material. Both moduli are shown to be linked together. As an application example, a two bi-phasic pattern is combined with some recent data from the literature to explore the broad predictive ability of the model. A full range of fibre volume fractions with fibre packing effects can be covered by the proposed method. Very simple analytical calculations make possible parametric studies less expensive than numerical or experimental characterizations.

1. Introduction

The automotive industry faces pressing environmental challenges aiming at offering sustainable transport solutions, mainly by ensuring a better fuel efficiency and reduced CO_2 emissions. Thanks to their microstructural and multi-scale modularity, fibre reinforced composite materials already provide durable, lightweight and high performance benefits in comparison with other competing (especially metal) parts. This trend drives composite materials into the heart of new material design. However, designing and testing new material solutions through an industrial production process is tedious and cost expensive. Alternatively, "*virtual testing*" can be used to get the effective properties of already known or targeted microstructures. However, in the particular case of unidirectionnal fibre reinforced composites, mechanical transverse properties (in contrast to longitudinal ones) are highly influenced by the fibre arrangement. This is particularly the case of the *transverse shear modulus* (μ_{23}^{eff}) and the *transverse bulk modulus* (or *plain strain bulk modulus* k_{23}^{eff}) which are among the most difficult parameters to measure experimentally. Therefore, a "realistic" microstructure model is required to understand how the microstructure affects the mechanical properties. In this work, an enhanced analytical micromechanical model is developed offering a fast and cost-effective solution to take into account the real microstructure as much as possible with only two parameters added to the overall volume fraction. For this purpose, a "*n*-phase" Generalised Self Consistent Scheme (GSCS) coupled with a Morphologically Representative Pattern (MRP) approach has been developed. The proposed analytical model is explained in section 2 and closed-form analytical expressions are given for μ_{23}^{eff} and k_{23}^{eff}. To handle complex morphologies while approaching as closely as possible real microstructures, section 3 addresses the problem of *Representative Volume Element* (RVE) using *Finite Element* microstructural *Models* (FEM) coupled with an *Integral Range* technique. Such numerical simulations help supplying the analytical implementation for a better description of interactions between the constituents and calibrate the only two morphological parameters. Numerical data referring to a UD glass fibre reinforced polyamide are provided with a known relative error of transverse elastic properties. Finally, those results are used in section 4 as an application example of the predictive ability of the model and a reverse engineering method is applied to obtain the most probable morphological parameters of the studied example.

2. Analytical modelling

The "*n*-phase" GSCS [1] is here coupled to a MRP approach [2] in order to take into account local morphological fluctuations within the microstructure of UD linear elastic multi-phased composite materials. The model was mainly inspired by three papers, [3] which follows a similar procedure in the case of a *Composite Sphere Assemblage* and [4, 5] which apply the same approach to the case of transport phenomenon. The present model consists consequentlty in an extension of the "*n*-phase" model for cylindrical inclusion embedded in an infinite matrix described in [1] to the case of (here two inverted) patterns submitted to homogeneous conditions (hydrostatic pressure or transverse shear $\underline{\underline{\varepsilon}}^0$) on its boundaries. For $i \in [1, 2]$, phase (i) has an internal radius $R_{i\text{-}1\,\lambda}$ and an external radius $R_{i\,\lambda}$ with $\lambda \in [1, 2]$. $(k_{23}^{(i)}, \mu_{23}^{(i)})$ denote respectively the plane strain bulk modulus and transverse shear modulus of phase (i).

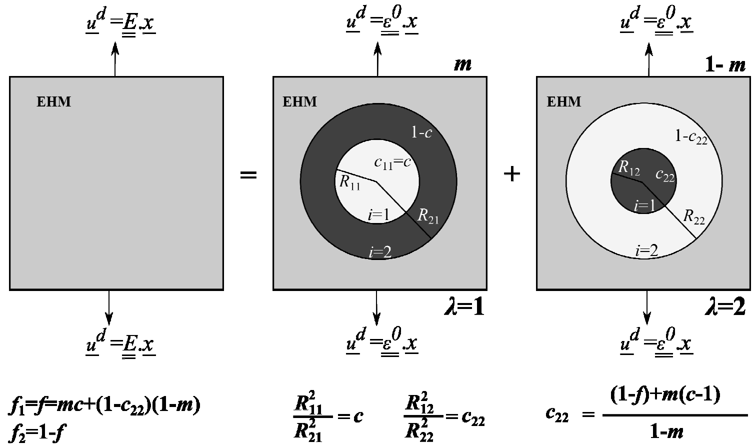

Figure 1. Two morphologically representative patterns.

The overall fibre volume fraction is f. The first pattern (λ=1) is called the "direct" pattern, the fibre is at the core of the pattern (i=1) and the volume fraction of this pattern is m. The second pattern (λ=2) is called the "inverted" pattern and the fibre (i=2) is now surrounding the matrix core. This second pattern allows to take into account local morphological fluctuations caused by fibre packings.

The average strain tensor in phase (i) inside the volume Ωi is given by (Eq. 1):

$$< \underline{\underline{\varepsilon}} >_{\Omega i} = \underline{\underline{\underline{\underline{A_i}}}} : \underline{\underline{E}} \tag{1}$$

and is used with the relationships between microscopic and macroscopic stress and strain fields to find (Eq. 2) where f_i denotes the volume fraction of phase (i).

$$\underline{\underline{\underline{\underline{C}}}}^{\text{eff}} = \sum_i f_i \underline{\underline{\underline{\underline{C}}}}^{(i)} : \underline{\underline{\underline{\underline{A_i}}}} \tag{2}$$

According to the boundary condition, (Eq. 2) leads to :

$$\begin{cases} k_{23}^{\text{eff}} & = \sum_{i=1}^{2} f_i k_{23}^{(i)} A_i \quad \text{when an hydrostatic pressure is imposed at infinity} \\ \mu_{23}^{\text{eff}} & = \sum_{i=1}^{2} f_i \mu_{23}^{(i)} A_i \quad \text{when a transverse shear is imposed at infinity} \end{cases} \tag{3}$$

In order to find A_i, the strain concentration tensor of the overall phase (i), three strain average calculations are necessary. A_i is found to depend on the strain concentration tensor of the phase (i) present in each pattern which is given in [6]. Finally, (Eqs. 4-6) provide the results as follows:

$$k_{23}^{\text{eff}} = \frac{A\mu_{23}^{\text{eff}} + B}{C\mu_{23}^{\text{eff}} + D} \tag{4}$$

where the constants A, B, C and D depend only on:

- the overall volume fraction of fibre f,
- the fibre volume fraction inside the “direct” pattern, c,
- the volume fraction of the “direct” pattern, m, and the transverse bulk and shear moduli of both phases, respectively $k_{23}^{(i)}$ and $\mu_{23}^{(i)}$

$$\mu_{23}^{\text{eff}} = \frac{\sum_{i=1}^{2} f_i \mu_{23}^{(i)} \sum_{\lambda=1}^{2} m_\lambda \left(\frac{P_\lambda^i(\mu_{23}^{\text{eff}})}{M_\lambda(\mu_{23}^{\text{eff}})} \right)}{\sum_{i=1}^{2} f_i \sum_{\lambda=1}^{2} m_\lambda \left(\frac{P_\lambda^i(\mu_{23}^{\text{eff}})}{M_\lambda(\mu_{23}^{\text{eff}})} \right)} \tag{5}$$

In (Eq. 5), P_λ^i is a linear function of μ_{23}^{eff} and M_λ is a quadratic function of μ_{23}^{eff} as defined below :

$$\begin{cases} M_\lambda(\mu_{23}^{\text{eff}}) & = A_\lambda \mu_{23}^{\text{eff}^2} + B_\lambda \mu_{23}^{\text{eff}} + C_\lambda \\ P_\lambda^i(\mu_{23}^{\text{eff}}) & = D_\lambda^i \mu_{23}^{\text{eff}} + E_\lambda^i \end{cases} \tag{6}$$

where A_λ, B_λ, C_λ, D_λ^i and E_λ^i depend on f, c, m, $k_{23}^{(i)}$ and $\mu_{23}^{(i)}$. It is worth noting that both effective moduli are linked together through the A_λ and B_λ constants which depend on k_{23}^{eff}. More details are given in [7].

3. Numerical modelling of the effective transverse shear modulus

The effective transverse shear modulus is known to be very difficult to be determined through mechanical testing. This modulus was studied for a long time with notable analytical results of Hermans [8] who derived the Hashin and Rosen relations by using the Kerner model [9] or the widely used result of the GSCS from Christensen and Lo [10]. However, despite constant prediction improvements , differences still remain between analytical results and experimental data. An intermediate step to understand the origin of these discrepancies is to use a “virtual characterization” via full field models. The main interest of full-field methods lies in an explicit consideration of a representative morphology of the material and without ambiguity, the control of the in-situ constituents properties. In a recent study [11], finite element simulations coupled with an integral range method were used to predict the elastic properties of glass fibre reinforced polyamide exhibiting a “random” arrangement of fibres. Thanks to the integral range statistical approach [12], properties can be obtained with a targeted relative error associating the size of the volume of the studied microstructure with the number of required realizations.

This approach gives the optimized size of the RVE (Representative Volume Element) for which “the parameters measured have a good statistical representativity”.

Let's consider a microstructure, of volume V, which fulfills the conditions of ergodicity and stationarity for a random variable $Z(x)$. $\langle Z\rangle$denotes the average of Z over n_r different (and independent) realizations and $D_z^2(V)$ the variance associated to $Z(x)$ on the volume V. The relative ϵ_{rel} and absolute ϵ_{abs} errors associated with the average value of Z are given by (Eq. 7):

$$\epsilon_{\mathrm{rel}} = \frac{\epsilon_{\mathrm{abs}}}{\langle Z\rangle} \quad \text{and} \quad \epsilon_{\mathrm{abs}} = \frac{2D_z(V)}{\sqrt{n_r}} \tag{7}$$

An estimation of the variance is given by [12] as $D_z^2(V) = K/V^{\alpha}$, K and α are determined thanks to a linear regression. The results obtained by [11] are given in (Table 1).

Table 1. Effective behaviour of a glass/polyamide UD composite material.

Parameter [Unit]	**Random FE calculus** $\epsilon_{\mathrm{rel}}^{\mathrm{max}} = 5\%$
μ_{23}^{eff} [MPa]	1 979
E_{11}^{eff} [MPa]	40 288
E_{22}^{eff} [MPa]	6 066
ν_{12}^{eff} [-]	0.28
ν_{23}^{eff} [-]	0.49

The transverse shear modulus has a mean value of 1979 MPa more or less 5% (relative error). It can be noticed that to minimize the relative error, either the number of realizations or the volume must increase. The application of the model (section 2) with the previous results is developed in section 4.

4. Application of the model to the case of a UD glass fibre reinforced polyamide

The material properties of the different phases of the studied composite are described in (Table 2). The average value of the effective transverse shear modulus μ_{23}^{eff} (1979 MPa), obtained with FE simulations and given in (Table 1) from [11], is provided with a relative error of 5% and varies between 1880 MPa and 2078 MPa.

Table 2. Materials data : Glass fibre and polyamide matrix.

Property	Notation [Unit]	Value
Global volume fraction of fibres	f [-]	0.57
Young modulus of the fibre	E_f [MPa]	70 000
Poisson's ratio of the fibre	ν_f [-]	0.22
Young modulus of the matrix	E_m [MPa]	1 350
Poisson's ratio of the matrix	ν_m [-]	0.4

The effective transverse bulk modulus k_{23}^{eff} is deduced from the following equation with the results of (Table 1):

$$k_{23}^{\text{eff}} = \frac{E_{22}^{\text{eff}}}{2\left(1 - \nu_{23}^{\text{eff}} - 2\dfrac{E_{22}^{\text{eff}} \nu_{12}^{\text{eff}\,2}}{E_{11}^{\text{eff}}}\right)} \tag{8}$$

Finally, (Eq. 8) leads to:

$$5593 \text{ MPa} < k_{23}^{\text{eff}} = 6236 \text{ MPa} < 6970 \text{ MPa} \tag{9}$$

The analytical results presented in section 2 are now used to estimate μ_{23}^{eff} and k_{23}^{eff} with the data provided in (Table 2) and for a given fibre volume fraction $f = 0.57$. The application of the model developed in section 2 leads to the effective transverse estimates plotted in (Fig. 2) for all the allowable m and c parameters.

The transverse effective shear (respectively bulk) modulus is given in terms of c for several values of m (dashed lines). The available values of the couple (c, m) are located between the two black envelope curves. Left (respectively right) triangles represent the minimum (respectively the maximum) values of c for each value of m as described in (Eq. 10). The red bottom point in (Fig. 2) represents the "n-phase" (GSCS) estimate with only one pattern (where $m = 1$ and $c = f = 0.57$). Search areas for m and c solutions correspond to the gray zone for k_{23}^{eff} and the dashed region for μ_{23}^{eff} (Fig. 2).

$$0 \le c_{min} = 1 + \frac{f-1}{m} < c < c_{max} = \frac{f}{m} \le 1 \tag{10}$$

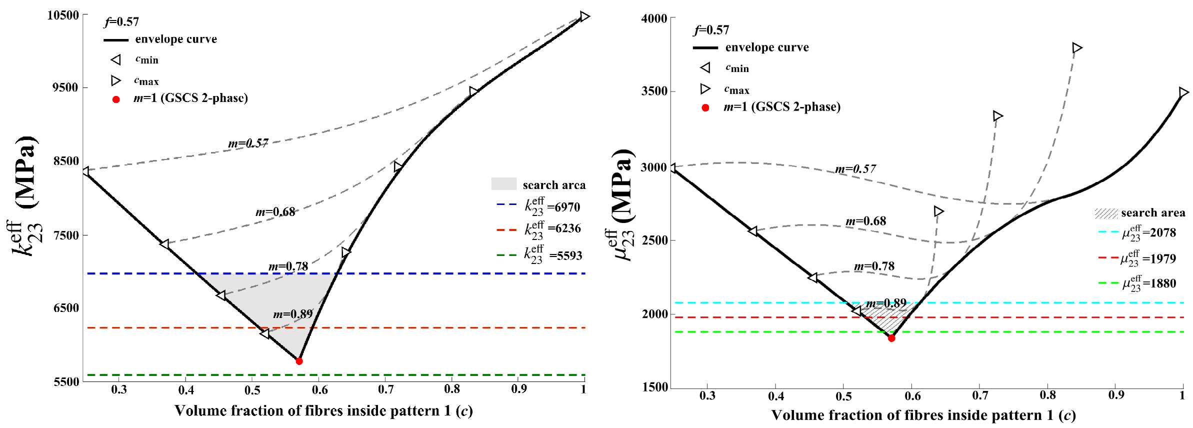

Figure 2. Effective bulk and shear moduli predicted thanks to the proposed analytical model.

The next step is to find the allowable couples (c, m) corresponding to an estimate of k_{23}^{eff} and μ_{23}^{eff} inside the search areas as shown in (Fig. 3).

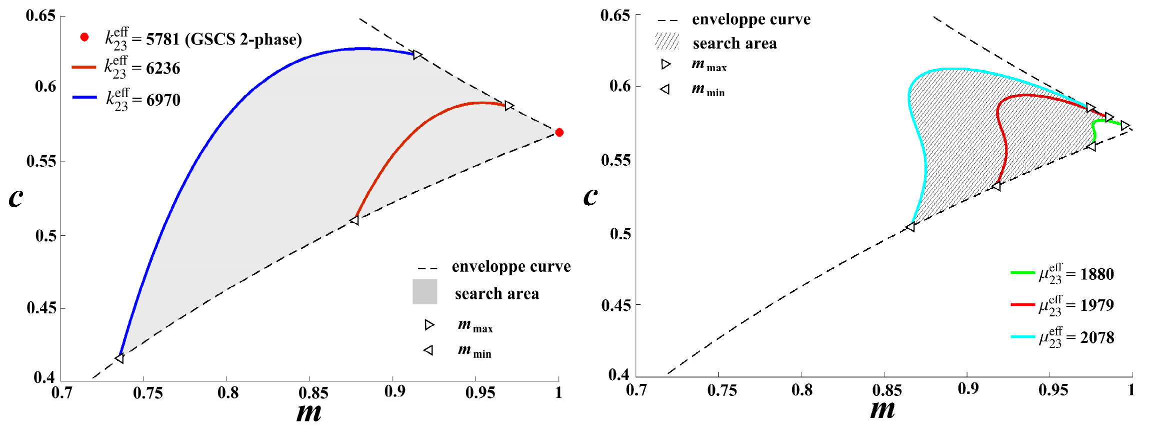

Figure 3. Allowable couples (c, m) for the given error intervals of k_{23}^{eff} in the left graph and of μ_{23}^{eff} in the right graph.

A common solution (c, m) for both k_{23}^{eff} and μ_{23}^{eff} is expected. For that purpose, the allowable solutions are shown in the same figure (Fig. 4).

According to (Fig. 4), the allowable region for k_{23}^{eff} includes entirely the allowable one for μ_{23}^{eff} . In that case, c varies from 0.5 to about 0.61 and m varies from 0.86 to about 0.99.

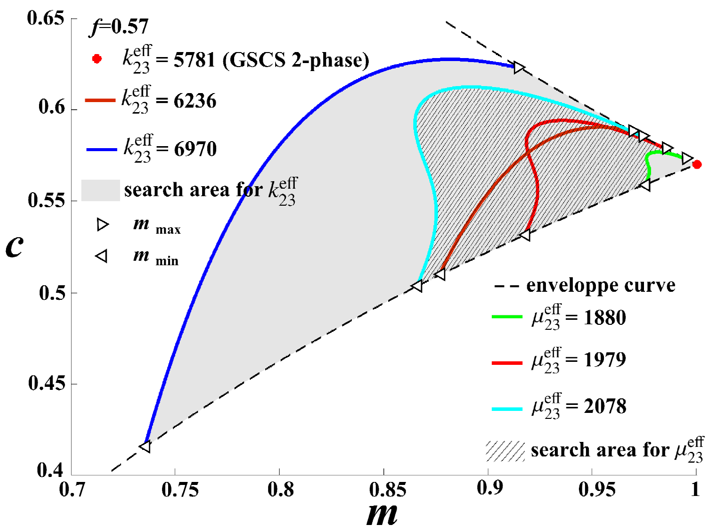

Figure 4. Comparison of the search areas of (c, m) for the given error intervals of both k_{23}^{eff} and μ_{23}^{eff}

5. Conclusion

A GSCS (Generalized Self Consistent Scheme) for transversely random UD composite has been investigated in this paper to take into account the influence of morphological fluctuations on the effective transverse elastic behaviour of such composite materials. For this purpose, a "*n*-phase" GSCS has been coupled to a MRP (Morphological Representative Pattern) approach leading to closed-form solutions depending on three parameters in the case of two patterns with two phases and presented in section 2. Some results of the numerical modelling presented in section 3 are given in section 4 in the case of glass fiber-reinforced polyamide and used to investigate the allowable values of the two parameters m and c for a given volume fraction f of fibers. It is worth noticing that an allowable area has been found (see (Fig. 4)) both for the effective bulk modulus k_{23}^{eff} and the shear modulus μ_{23}^{eff}. This area will be reduced thanks to a greater number of full-field simulations. The aim of this study is to provide a simple analytical model to get m and c calibrated by FEM for various morphological configurations. Finally it will be possible to carry on parametric studies without tedious numerical simulations to obtain for instance targeted microstructures or to take into account morphological fluctuations.

Acknowledgments

This study is part of a program supported by Michelin Company (France). Fruitful discussions with R. Bruant and A. Mbiakop-Ngassa , partners of this program, are gratefully acknowledged.

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