Magnetic eigenmodes in the plesio-geostrophic model

Jingtao Min, Andrew Jackson, Stefano Maffei, Institute of Geophysics, ETH Zurich

Abstract: The most dominant contribution to the temporal variations of the geomagnetic field at long time scales comes from the electric current generated by the flow of the electrically conductive fluid in the outer core. Rapid rotation of the Earth places the outer core fluid near the geostrophic force balance, organising the flow into columnar structures. Making use of the columnar flow ansatz, the recent plesio-geostrophic (PG) model shows that the ideal magnetohydrodynamic (MHD) equations in 3-D sphere can be reduced to equations on 2-D manifolds, allowing more efficient computation in more realistic parameter space. We present here the magnetic eigenmodes using the PG equations, linearized around certain background fields.

The bigger picture: Geomagnetic data assimilation, and why we want a reduced-dimensional model

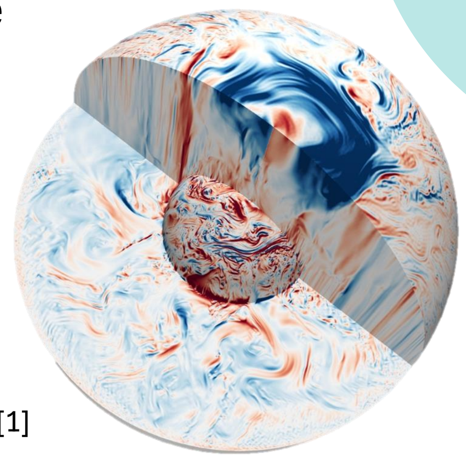
- The major part of Earth's magnetic field (*main field*) is generated by the dynamo process in the fluid outer core (*geodynamo*). Conversely, magnetic field observation is the most important of the few observations that yield information on the underlying core dynamics.
- **Geodynamo modelling** calculates evolution of magnetic field based on initial condition (IC), boundary condition (BC) and physical properties of the deep interior.
- **Geomagnetic data assimilation** tries to retrieve information on the initial state, BCs and physical parameters from magnetic observations, combined with a geodynamic model.
- 3-D simulations are desirable, but...
  - Computationally very expensive (esp. repeated evaluations)
  - Use parameters that are far away from the realistic Earth

Param.	Earth	Model
Ekman $\mathrm{E} = \frac{\nu}{\Omega R^2}$	10^{-15} - 10^{-14}	10^{-7}
Magnetic Prandtl $\mathrm{Pm} = \frac{\nu}{\eta}$	10^{-6} - 10^{-5}	10^{-1} - 10^{3}

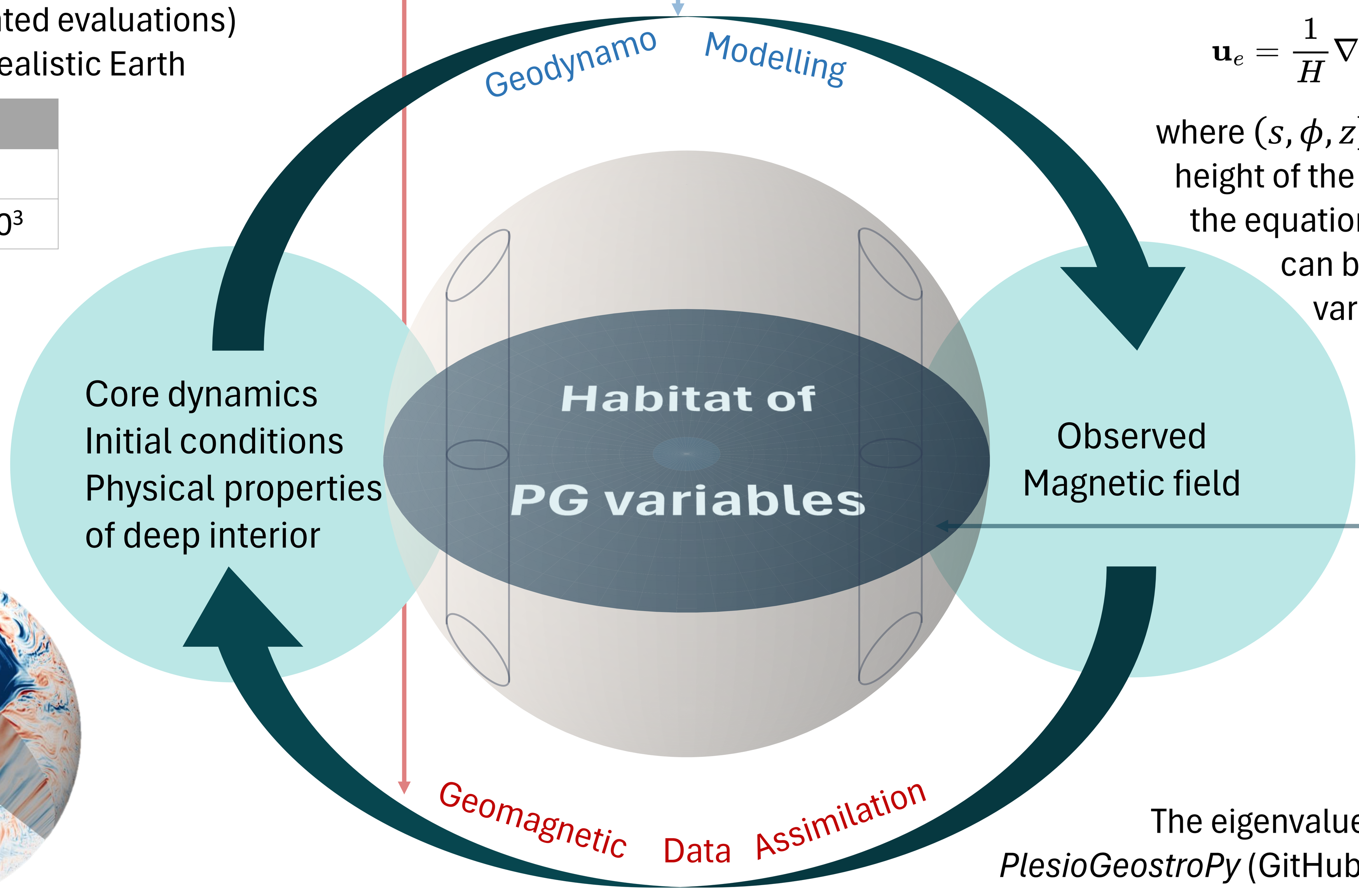
- Reduced-dimensional models: efficiency + realistic parameters

How? Geostrophic balance

- Earth's core is in the *rapid rotating regime*
- Leading force balance: geostrophic balance
- Flow at moderate scales are organized into columnar structures invariant along the rotation axis
- Endmember case: Taylor Proudman theorem



Right: azimuthal velocity, 3-D simulation from [1]



The Plesio-geostrophic (PG) model

Governing equations: Navier Stokes (Coriolis + Lorentz forces) + magnetic induction equation

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} + \frac{2}{\mathrm{Le}}\hat{\mathbf{z}}\times\mathbf{u} = -\nabla p + (\nabla\times\mathbf{B})\times\mathbf{B} + \frac{\mathrm{Pm}}{\mathrm{Lu}}\nabla^2\mathbf{u}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla\times(\mathbf{u}\times\mathbf{B}) + \frac{1}{\mathrm{Lu}}\nabla^2\mathbf{B}$$

The dimensionless groups Lehnert number (Le), Lundquist number (Lu) and magnetic Prandtl number (Pm) are defined as

$$\mathrm{Le} = \frac{\tau_\Omega}{\tau_\eta} = \frac{\mathcal{B}}{\sqrt{\rho\mu_0}\Omega R} \qquad \mathrm{Lu} = \frac{\tau_\eta}{\tau_A} = \frac{\mathcal{B}R}{\sqrt{\rho\mu_0}\eta} \qquad \mathrm{Pm} = \frac{\nu}{\eta}$$

Here we consider the ideal limit where $\mathrm{Lu}\to+\infty$. Using the columnar ansatz [2]:

$$\mathbf{u}_e = \frac{1}{H}\nabla\times(\psi\hat{\mathbf{z}}), \quad u_z = z\beta u_s, \quad \beta = \frac{1}{H}\frac{dH}{ds}$$

where (s,ϕ,z) cylindrical coordinates, and $H=\sqrt{1-s^2}$ the half height of the cylinder, and by applying even / odd axial integrals, the equations of velocity and magnetic fields in the 3-D sphere can be reduced *exactly* to a set of equations of 14 scalar variables that all live in the 2-D equatorial plane [3, 4]:

$$\psi, \overline{B_s^2}, \overline{B_\phi^2}, \overline{B_sB_\phi},$$
$$\widetilde{B_sB_z}, \widetilde{B_\phi B_z}, \widetilde{zB_s^2}, \widetilde{zB_\phi^2}, \widetilde{zB_sB_\phi},$$
$$B_s^e, B_\phi^e, B_z^e, B_{s,z}^e, B_{\phi,z}^e$$

$$\bar{f} = \int_{-H}^{+H} f(z)dz, \quad \widetilde{f} = \int_{-H}^{+H}\mathrm{sgn}(z)f(z)dz$$

Numerical scheme

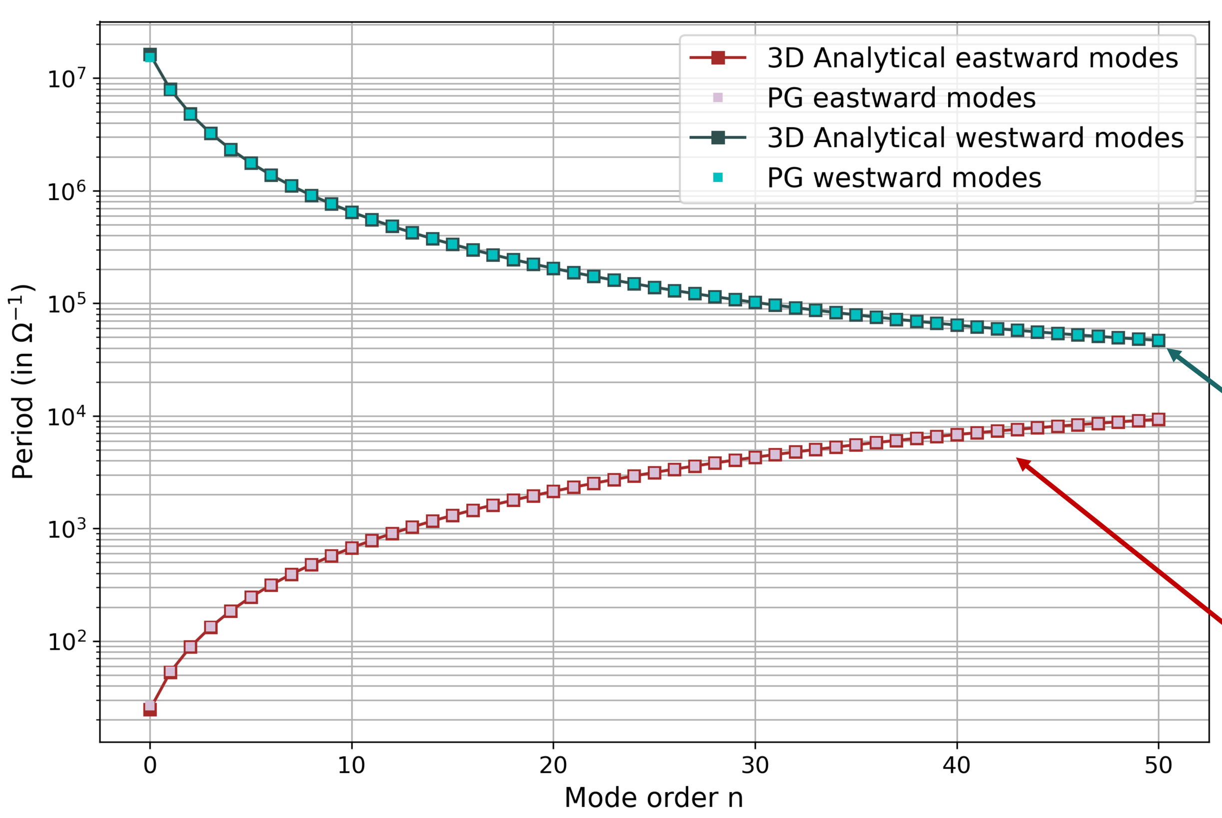
The eigenvalue problems are solved using the fully spectral code *PlesioGeostroPy* (GitHub public library), with Fourier basis in the azimuthal direction, and Jacobi polynomials with prefactors satisfying regularity conditions at the origin.

Eigenmodes under the Malkus field

Under the Malkus background field:

$$\mathbf{B} = s\hat{\phi}$$

the ideal 3-D system admit closed-form eigenvalue solutions [4]. Our numerical eigenvalue solution to the PG system under the same background field shows remarkable agreement with the analytical eigenfrequencies in 3D.



Eigenperiods of the PG magnetic eigenmodes for azimuthal wavenumber m=3, under the Malkus background field, as compared to the analytical solution for the ideal 3-D system. The eastward modes (red / pink) are inertial-mode-like fast modes, while the westward modes (blue / cyan) are magnetostrophic-mode-like slow modes.

Eigenmodes under a poloidal dipolar field

Under a poloidal dipolar background field:

$$\mathbf{B} = \nabla\times\nabla\times\left(r(3r^2-5)Y_1^0(\theta,\phi)\mathbf{r}\right)$$
$$= -6sz\hat{\mathbf{s}} + (12s^2+6z^2-10)\hat{\mathbf{z}}$$

There is no known analytical 3-D eigenmode in this case; the results are compared to the numerical 3-D eigenmode solutions [5]. The parameters are given below:

Params	Lehnert Le	Lundquist Lu
PG model	10^{-4}	$+\infty$
3-D model	10^{-4}	5×10^4

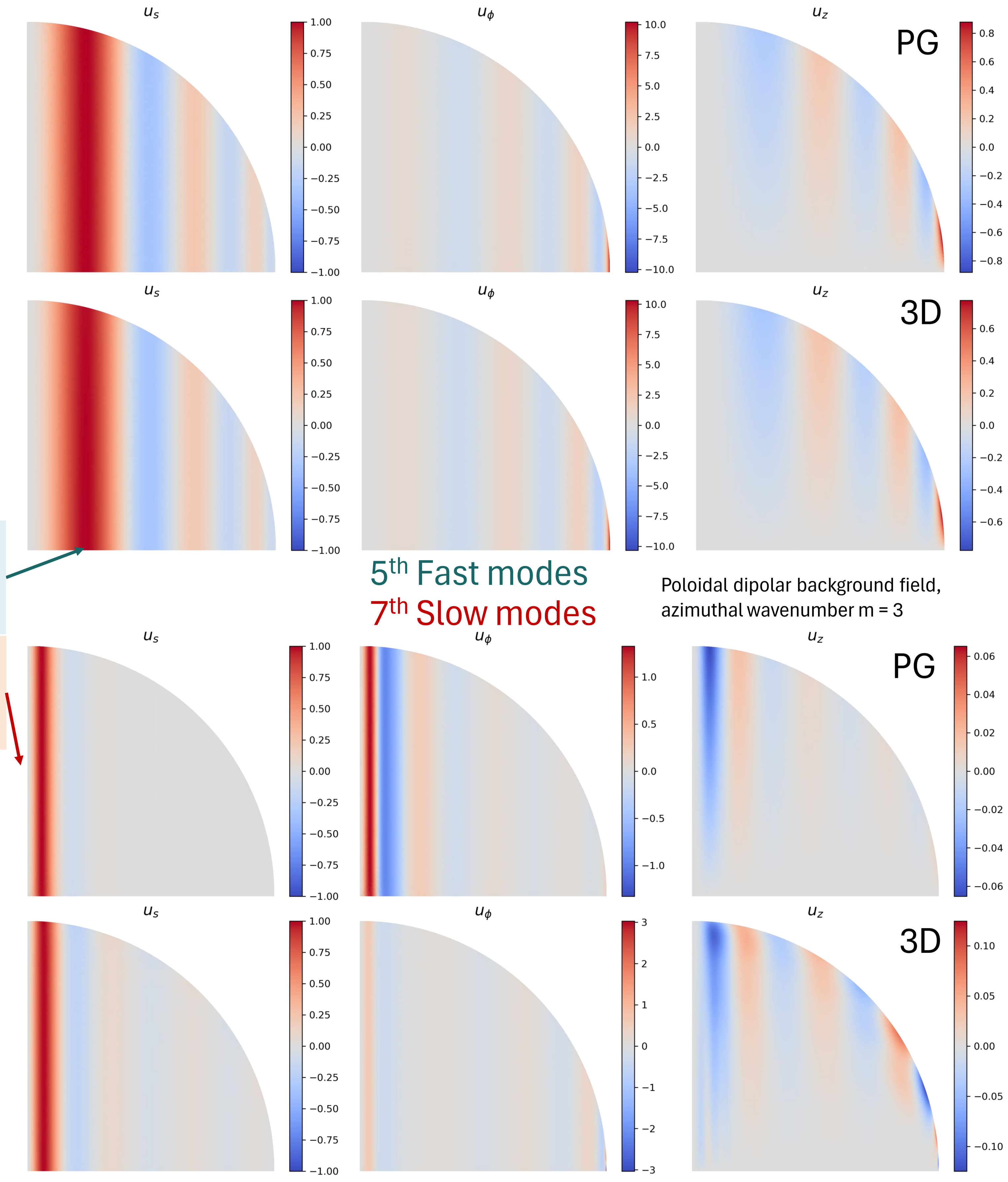
Converged columnar eigenmodes are found in both fast and slow branches.

The *inertial-mode-like* branch consists of eigenmodes (*fast modes*) dominated by inertial effects.

The *magnetostrophic-mode-like* branch consists of eigenmodes (*slow modes*) dominated by magnetostrophic balance.

Eigenvalues	Fast (upper right)	Slow (bottom right)
PG $[T_a^{-1}]$	-259.4i	+5.331i
3-D $[T_a^{-1}]$	-256.5i – 4.5e-4	+0.3323i – 0.02647

The geometry of the PG eigenmodes are compared with the corresponding 3-D modes. The distribution of the field shows good agreement. However, the eigenvalues of the slow modes do not match between PG and 3D. The trend also does not agree with [6]. The problem remains to be answered.



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Conclusions and outlook

The successful calculation of magnetic eigenmodes tests the validity of the PG model. This is the first of several steps towards a framework in plan that utilises the PG model as the dynamical core for geomagnetic data assimilation. Stay tuned!

Stay tuned!

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