The Zipf's law as the unique equilibrium in systems of cities

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Abstract

It is well known that the standard system of cities suffers from two major drawbacks. A spatial equilibrium i)- is not unique and ii)- cannot reproduce the basic fact that big cities follow an exact Zipf's law under credible conditions. To fix this long-lasting puzzle, I build a new theory of systems of cities that generates three key results. First, the standard indirect utility function à la Henderson is obtained using hyperbolic preferences. Second, a Zipf's law can hold in the model, depending on the value of TFP parameters. Third, a unique spatial equilibrium can be obtained, depending on the value of agricultural rents.

JEL Classification: R12, D62

Keywords: Systems of cities, uniqueness, Zipf's law

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"One well-known striking regularity in the size distribution of cities is that it is roughly log-normal, with an upper tail that is statistically indistinguishable from a Pareto distribution with unitary shape parameter: Zipf's law holds for (large) cities" in Behrens and Robert-Nicoud (2015)

"To solve the equilibrium selection problem, the literature has often relied on the existence of large-scale, competitive land developers. When sites are homogeneous, the equilibrium with land developers is both unique and (generally) efficient, arguably two desirable properties" in Behrens and Robert-Nicoud (2015)

1 Introduction

Many empirical studies document that the size distribution of large cities follows a Zipf's law. This is particularly acute for the US, and this phenomenon is also observed in many other countries in the world (see, among others, Gabaix (1999), Eeckhout (2004) and Gabaix and Ioannides (2004)).

Unfortunately, the standard system of cities à la Henderson (widely used in the literature) cannot reproduce this stylized fact. This long-lasting puzzle is explained by two reasons. On the one hand, the direct connection between a spatial equilibrium and the Zipf's law is not established under plausible conditions. On the other hand, due to the presence of agglomeration economies and congestion costs, the indirect utility function is non-monotonic. This creates multiplicity. In particular, it has been proven that there exists a continuum of (stable) equilibria (see Henderson (1974) and Behrens and Robert-Nicoud (2015)). In sum, the standard system of cities à la Henderson seems to be incapable of generating a unique spatial equilibrium that follows a Zifp's law.

In this article, I qualify this statement by developing a new theory of systems of cities. The framework considered here encapsulates the following key ingredients. The geography is composed of a finite set of cities. Following Behrens and Robert-Nicoud (2015) and Albouy et al. (2019), cities are heterogeneous according to an endowment of amenities. The economy is also populated by homogeneous workers. These workers are endowed with hyperbolic preferences (see Mossay and Picard (2011), Blanchet et al. (2016) and Boitier (2018)) and choose in which city to live by maximizing an indirect utility function.[1] The latter only depends on the level of amenities and on the density of workers in a given city. The presence of the density of workers in a given city encapsulates both agglomeration economies and congestion costs.

Armed with this new framework, I derive three results that qualify convention wisdom. First, I provide a new micro-foundation for the standard indirect utility function (see equation (1)) used in the literature manipulating hyperbolic preferences.

[1] Hyperbolic preferences are not related to hyperbolic discounting.

Second, I demonstrate that a spatial equilibrium can be unique, depending on the value of agricultural rents. When agricultural rents are null/small, the utility level exhibits an inverted $U-$shape curve. This creates multiplicity as in standard theoretical models. When agricultural rents are high enough, the indirect utility function is no longer a single-peaked function, and becomes monotonically decreasing with population. This ensures that a unique spatial equilibrium exists. I also show that the condition ensuring uniqueness is credible and not demanding.

Third, I find that the new framework can generate an exact Zipf's law, depending on the TFP distribution. Interestingly, I find that the shape of the TFP distribution is not detrimental to generate a spatial equilibrium that follows a Zipf's law. Notably, I obtain that a linear distribution of TPF parameters can deliver the shape of the Zipf's law (which is very skewed). This suggests that the new theory of system of cities endogenously leads to a Zipf's law.

These results contribute to the design of systems of cities according to two main dimensions. On the one hand, I unveil a new use for hyperbolic preferences. Hyperbolic preferences were introduced by Mossay and Picard (2011). Blanchet et al. (2016) consider hyperbolic preferences in a complex model of social interactions. Boitier (2018) highlights the interest of these preferences in urban search models. In this new framework, hyperbolic preferences permit to derive a closed-form solution similar to the standard utility function à la Henderson. More globally, I highlight a new link between models with endogenous land consumption and models without endogenous land consumption. Namely, the standard model with no endogenous land consumption but with "power" transportation costs (i.e, equation (1)) is equivalent to a model with endogenous land consumption and hyperbolic preferences but without transportation costs (i.e, equation (10)).

On the other hand, I show that a unique Zipf's equilibrium is available in a system of cities à la Henderson (see Kanemoto (1980), Fujita (1989) and Behrens and Robert-Nicoud (2015) for a survey of the literature). Some frameworks can generate such a law. This is the case of random growth models (see for instance Gabaix (1999), Eeckhout (2004), Rossi-Hansberg and Wright (2007), Hsu (2012) and Lee and Li (2013)). But random growth models are pure statistical phenomena without any underlying economic theory. By contrast, it is well acknowledged that the standard theoretical framework suffers from two major drawbacks: a spatial equilibrium is not unique and cannot reproduce the basic fact that big cities follow an exact Zipf's law under credible conditions. The present article fixes this puzzle by highlighting the interplay of (hyperbolic) preferences, agricultural rents and TFP parameters. Notably, I show that the Zipf's law holds even if TFP parameters do not follow a Zipf's law. This strongly contrasts with random growth models where the size distribution of cities inherits the properties of the distribution of TPF parameters. In random growth models, TPF parameters must follow a Zipf's law to lead to a size distribution of cities that is a Zipf's law. As a consequence, the ability of these statistical models to generate the features of the data is fairly exogenous. Here, this is not the case. The new framework endogenously generates a Zipf's law.

Last, it is worth noting that having a unique spatial equilibrium is desirable. This is valuable as it permits to overcome the equilibrium selection problem. As a consequence, the present framework could be useful in studies wishing to obtain tractable results. In particular, the present article offers a well-defined setting such that it becomes possible to carry out a robust comparative statics analysis, to measure accurately welfare gains, etc.

The article is structured in the following manner: Section 2 motivates the topic, Section 3 develops the model and Section 4 provides the conclusions.

2 Motivations and puzzle

In this section, I motivate the framework of Section 3. Notably, I clarify the notion of a spatial equilibrium, and I highlight the two shortcomings of the literature.

2.1 Indirect utility function and spatial equilibria

The standard system of cities is based on the following indirect utility function for workers living in city $c \in \mathcal{C} = \{1, ..., C\}$ (see Behrens and Robert-Nicoud (2015) for more details and for a derivation):

$$v(c, \ell_c) = \mathbb{A}_c \ell_c^{\epsilon} - \ell_c^{\gamma} \tag{1}$$

where $\mathbb{A}_c$ is a TFP parameter for city c, ℓ_c is the mass of workers living in city c, $\epsilon > 0$ is the elasticity of agglomeration economies with respect to population and $\gamma > 0$ is the elasticity of congestion costs with respect to population. The vector of TFP parameters $\mathbb{A} = (\mathbb{A}_1, ..., \mathbb{A}_c, ..., \mathbb{A}_C)$ encompass the heterogeneity of locational fundamentals including natural endowments, transportation infrastructures, historical amenities, etc.

This vector is key. For instance, Ellison and Glaeser (1999) document that differences in TFP parameters explain about one-fifth of the observed geographical concentration. Agglomeration economies regroup scale economies and information spillovers, and are expected to improve cities size (see Duranton and Puga (2004)). Congestion forces capture transportation costs and land prices, and are expected to limit cities size (see Fujita (1989)). Technically, to obtain the RHS of equation (1), the literature traditionally assumes a system of cities with internal structures and specific transportation costs. This means that each city is a monocentric city with a continuum of locations, and workers face "power" transportation costs ($\tau(.) = .^{\gamma}$ with τ being the transportation cost function) to go to the center of the city. Similarly, to obtain the LHS of equation (1), the literature assumes that wages are driven by agglomeration forces (see Duranton and Puga (2004)).

Equipped with equation (1), a spatial equilibrium is defined as follows in the literature.

Definition 1 *A spatial equilibrium denoted by* $\ell^* = (\ell_1^*, ..., \ell_c^*, ..., \ell_C^*) \in \mathcal{M}(\mathcal{C})$ *verifies the following:*

$$\begin{cases} \mathbb{A}_c \ell_c^{*\epsilon} - \ell_c^{*\gamma} \leq v^* & \textit{for almost every } c \in \mathcal{C} \\ \mathbb{A}_c \ell_c^{*\epsilon} - \ell_c^{*\gamma} = v^* & \textit{for almost every } c \in \mathcal{C} \textit{ such that } \ell_c^* > 0 \end{cases}$$

where the set $\mathcal{M}(\mathcal{C})$ *is defined as:*

$$\mathcal{M}(\mathcal{C}) = \left\{ \ell = (\ell_1, ..., \ell_c, ..., \ell_C) \in [0, L]^C : \sum_{c \in C} \ell_c = L \right\}$$

$L > 0$ is the total mass of workers in the system of cities and $v^* \in \mathbb{R}$ is a constant capturing the utility reached by workers in equilibrium. Note that if $L = 1$ then $\mathcal{M}(\mathcal{C})$ collapses to the standard simplex:

$$\mathcal{M}(\mathcal{C}) = \left\{ \ell = (\ell_1, ..., \ell_c, ..., \ell_C) \in [0, 1]^C : \sum_{c \in C} \ell_c = 1 \right\}$$

The notion of a spatial equilibrium is straighforward. It corresponds to a situation in which each worker receives the same utility level wherever it locates. In such a configuration, unilateral deviations of strategies are impossible.

The literature also considers a restrictive version of Definition 1. This is done as follows.

Definition 2 *A "full-support" (hereafter FS) spatial equilibrium* $\ell^* = (\ell_1^*, ..., \ell_c^*, ..., \ell_C^*) \in \mathcal{M}(\mathcal{C})$ *verifies the following:*

$$\begin{cases} \mathbb{A}_c \ell_c^{*\epsilon} - \ell_c^{*\gamma} = v^* & \textit{for all } c \in \mathcal{C} \\ \ell_c^* > 0 & \textit{for all } c \in \mathcal{C} \\ \sum_{c \in C} \ell_c^* = L \end{cases}$$

The support of ℓ is said to be "full" if each country attracts at least some workers: $Supp(\ell) = \mathcal{C}$. This is empirically warranted when the system of cities discribes big cities only (or the upper tail of the size distribution of cities). In that case, C is expected to be "small": $C \in \{5, 10, 20, 50, 100\}$. A FS spatial equilibrium is also a common object in theoretical urban models. For example, the model considered by Behrens and Robert-Nicoud (2015) generates a FS spatial equilibrium. Similarly, a spatial equilibrium given by a random utility model (see McFadden (1974)) is a FS spatial equilibrium (see Behrens and Murata (2021)).

2.2 Zipf's law

Rearranging equilibrium conditions in Definition 1 or in Definition 2 yields:

$$\ell_c^* = \left(\mathbb{A}_c - \frac{v^*}{\ell_c^{*\epsilon}}\right)^{\frac{1}{\gamma-\epsilon}}$$

Behrens and Robert-Nicoud (2015) observe that if the following assumption is complied for a given city c:

$$\lim_{\ell_c^* \to \infty} \frac{v^*}{\ell_c^{*\epsilon}} = 0 \tag{2}$$

then ℓ_c^* is proportional to $\mathbb{A}_c$:

$$\ell_c^* \approx \mathbb{A}_c^{\frac{1}{\gamma-\epsilon}}$$

This means that the shape of city c inherits the properties of the TFP distribution. In turn, if $\mathbb{A}_c$ follows a Zipf's law then ℓ_c^* follows a Zipf's law. However, assumption (2) is not satisfactory for two reasons. First, assumption (2) is local, it concerns city c only. The assumption does not guarantee that the whole distribution follows a Zipf's law. Put differently, condition (2) says that, to obtain an exact Zipf's law, each size of cities must be infinite in equilibrium: $\ell_c^* \to \infty$ for all c, which seems to be untenable. Second, assumption (2) rarely occurs. For example, the assumption is easily violated if $L = 1$ (or L small), a conventional value set in many calibrations. In other worlds, the standard system of cities à la Henderson has difficulty to reproduce the fact that big cities follow an exact Zipf's law.

2.3 Multiplicity

Equation (1) postulates an inverted U-shape of utility levels against city size. See Figure 1 for an example in which $C = 3$ and such that $\mathbb{A}_1 > \mathbb{A}_2 > \mathbb{A}_3$. Because the utility function of workers is a a single-peaked function, uniqueness of a spatial equilibrium does not hold. Rather, it is readily verified that there exists a continuum of (stable) equilibria that can be Pareto-ranked (see Henderson (1974) and Behrens and Robert-Nicoud (2015)).[2]

Unfortunately, the multitude of equilibria is harmful by itself. Uniqueness is desirable as it eliminates the "equilibrium selection problem". In Henderson economies, the selection of an equilibrium is a formidable issue, and the precise spatial equilibrium to be selected is totally undetermined. Another shortcoming lies in the great diversity of the features of equilibria. Differences may be substantial. Equilibria may distinguish themselves in terms of sites (fewer or more cities), in terms of city sizes (smaller or larger city sizes) and in terms of utility levels (lower or higher utilities). For example, Henderson and Becker (2000) develop and calibrate a system of cities with homogeneous sites. They show that

[2]Stability necessarily requires $\gamma > \epsilon$.

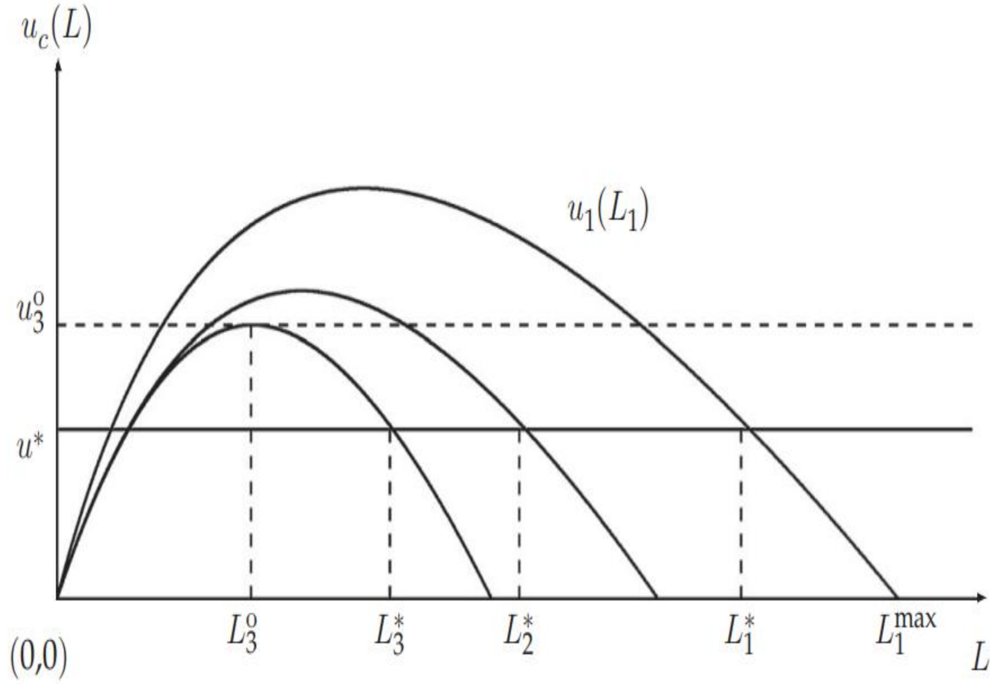

Figure 1: Multiplicity in systems of cities, from Behrens and Robert-Nicoud (2015)

stable spatial equilibria lie in $[\ell_{min}, \ell_{max}]$. Under different conventional calibrations, they find that ℓ_{max} is 10-75 times ℓ_{min}. As a consequence, this lack of determinacy prevents the possibility to carry out a robust comparative statics analysis, to measure accurately welfare gains, etc.

2.4 Conclusions and statements of main results

In a nutshell, the inability of the canonical model of systems of cities to generate a unique Zipf's law is well-established, which constitutes a valuable puzzle. The natural question to ask is to know if there exists another framework that mimics the properties of the standard model (i.e. an alternative framework that generates equation (1) as an outcome) but that leads to a unique Zipf's law. Hereafter, I develop such a new model. In particular, I build a framework that offers a new micro-foundation for equation (1). I then show that the spatial equilibrium in this new framework can be unique, depending on the values of agricultural lands, and can be a Zipf's law, depending on the distribution of TFP parameters.

3 Model

In Section 3.1, I describe the setup. In Section 3.2, I clarify the notion of a spatial equilibrium. In Section 3.3, I display the results.

3.1 Environment

In what follows, I describe the geography and the characteristics of workers.

3.1.1 Geography

The economy consists in a system of C big cities denoted by $c \in \mathcal{C} = \{1, ..., C\}$. Each city is endowed with a level of amenities expressed by $\mathbb{A} = (\mathbb{A}_1, ..., \mathbb{A}_c, ..., \mathbb{A}_C)$. $\mathbb{A}$ is ordered: $\mathbb{A}_1 > ... > \mathbb{A}_c > ... > \mathbb{A}_C$: the city 1 ($C$) is the most (least) attractive in the system.

3.1.2 Workers' problem

The economy is populated by a mass L of homogeneous workers. Workers living in city $c \in \mathcal{C}$ are endowed with hyperbolic preferences such that:

$$u(z_c, h_c) = z_c - \frac{\phi}{h_c^\gamma}, \quad \phi, \gamma > 0$$

for all c in $\mathcal{C}$. z is the amount of composite consumer good used as numéraire, h is the lot size of houses and ϕ and γ are two parameters that capture preference for land. Another meaning for γ is done further.

Hyperbolic preferences were introduced by Mossay and Picard (2011). Boitier (2018) highlights the interest of these preferences in urban search models. More globally, hyperbolic preferences are quasi-linear preferences. This implies that the income effect is ineffective in the model. However, the advantages of manipulating such preferences are multiple. First, it permits to derive a closed-form solution similar to equation (1). Second, the obtained reduced form enables me to thoroughly delimit the role of agricultural rent in urban formation. Third, the determination of equilibrium is easy. Conditions for uniqueness are simple and analytical. Last, it is worth noting that the preferences does not determine the nature of the results. It only gives tractability.

In addition, the budget constraint of the workers living in city $c \in \mathcal{C}$ is given by:

$$z_c + R_c h_c = \omega_c$$

for all c in $\mathcal{C}$. ω is the worker's wage and R is the rent per unit of land (paid to absentee landlords).

In line with Behrens and Rober-Nicoud (2015) and Albouy et al. (2019), wages are determined by firms that compete for labor and that have access to the amenity $\mathbb{A}$:

$$\omega_c = \mathbb{A}_c \ell_c^\epsilon, \quad \epsilon > 0$$

where ℓ_c is the mass of workers living in $c \in \mathcal{C}$. Production exhibits external economies of scale that are captured by ϵ the elasticity of agglomeration economies with respect to population.

Within this framework, workers play the following two-step game:

Step 1. They choose a single city to live.

Step 2. They choose how much land to consume.

Therefore, the entire program of a worker is given as follows:

$$\max_{c,h_c} \left\{ \mathbb{A}_c \ell_c^\epsilon - R_c h_c - \frac{\phi}{h_c^\gamma} \right\} \tag{3}$$

after some algebra.

3.2 Equilibrium

The worker's problem is solved by backward induction.

3.2.1 Equilibrium in Step 2

For $c \in \mathcal{C}$ fixed, maximizing (3) with respect to h leads to the following solution.

Proposition 1 *The equilibrium housing demand in city c is given by:*

$$h_c^* = \left(\frac{\gamma \phi}{R_c} \right)^{\frac{1}{\gamma+1}} \tag{4}$$

that gives:

$$z_c^* = \mathbb{A}_c \ell_c^\epsilon - (\gamma \phi)^{\frac{1}{\gamma+1}} R_c^{\frac{\gamma}{\gamma+1}}$$

The Marshallian demand (4) is independent of the revenue of workers. It only relies positively on preference for land (γ and ϕ) and negatively on the rent per unit of land R. Obviously, the reason for that is that preferences are quasi-linear.

3.2.2 Equilibrium in Step 1

To pin down the location of households across cities, it is possible to use the bid rent theory. The latter consists in determining the maximum rent that a worker would pay for living in a given city. The bid rent function of the workers residing in city $c \in \mathcal{C}$ is given by:

$$\Psi(c, v^*) = \left[\frac{\mathbb{A}_c \ell_c^{\epsilon} - v^*}{(\gamma + 1)(\gamma\phi)^{\frac{1}{\gamma+1}}} \right]^{\frac{\gamma+1}{\gamma}}$$

where v^* corresponds to the equilibrium indirect utility function. Then, land is allocated to the highest bid rent:

$$R_c^* = max \{\Psi(c, v^*), R_A\}$$

As a consequence, a spatial equilibrium is defined in the following manner.

Definition 3 *A spatial equilibrium* (R^*, ℓ^*) *verifies the following:*

$$R_c^* = max \{\Psi(c, v^*), R_A\} \tag{5}$$

$$u(z_c^*, h_c^*) = v^* \tag{6}$$

$$h_c^* \ell_c^* = 1 \tag{7}$$

$$\sum_{c \in \mathcal{C}} \ell_c^* = L \tag{8}$$

$$z_c^* > 0 \tag{9}$$

for all $c \in \mathcal{C}$.

Equation (5) demonstrates that land is allocated to the highest bid rent. Equation (6) is the standard non-arbitrage condition stating that all agents reach the same utility level. (7) satisfies the land market equilibrium, (8) meets the total population constraint and (9) imposes strictly positive consumption in equilibrium.

3.2.3 Rewriting equilibrium in Step 1

After simple algebra, I find the following.

Proposition 2 *An equilibrium in Step 1 verifies the following:*

$$v^* = \mathbb{A}_c \ell_c^{*\epsilon} - (\gamma + 1)\phi \ell_c^{*\gamma}, \quad \forall c \in \mathcal{C} \tag{10}$$

$$\left(\frac{R_A}{\gamma\phi} \right)^{\frac{1}{\gamma+1}} \leq \ell_c^* < \left[\frac{\mathbb{A}_c}{\gamma\phi} \right]^{\frac{1}{\gamma-\epsilon}}, \quad \forall c \in \mathcal{C}$$

$$\sum_{c \in \mathcal{C}} \ell_c^* = L$$

Several comments are in order.

First, γ has now an accurate definition. It is called the "size elasticity of urban costs", and corresponds to the elasticity of congestion costs (here land rents costs) with respect to population. As usual, I assume that $\gamma > \epsilon$ (see Behrens and Robert-Nicoud (2015)).

Second, the fact that v^* depends on ℓ^* translates the existence of a trade-off between agglomeration and dispersion forces. Here, agglomeration forces are positive urban externalities encompassing external effects in wages, and dispersion forces are negative urban externalities capturing residential costs, and more globally, living costs.

Third, equation (10) qualitatively corresponds to equation (1). In so doing, the present article provides a new micro-fundation manipulating hyperbolic preferences. More globally, Proposition 2 establishes a new link between models with endogenous land consumption and models without endogenous land consumption. Namely, the standard model with no endogenous land consumption but with "power" transportation costs (i.e, equation (1)) is equivalent to a model with endogenous land consumption and hyperbolic preferences but without transportation costs (i.e, equation (10)).

Last, a new feature of a spatial equilibrium is that the size of each city is bounded. Notably, the lower-bound $\left(\frac{R_A}{\gamma\phi}\right)^{\frac{1}{\gamma+1}}$ is obtained as follows: $\Psi(c, v^*) \geq R_A \Leftrightarrow \left(\frac{R_A}{\gamma\phi}\right)^{\frac{1}{\gamma+1}} \leq \ell_c^*$. The upper-bound $\left[\frac{\mathbb{A}_c}{\gamma\phi}\right]^{\frac{1}{\gamma-\epsilon}}$ is obtained as follows: $z_c^* > 0 \Leftrightarrow \ell_c^* < \left[\frac{\mathbb{A}_c}{\gamma\phi}\right]^{\frac{1}{\gamma-\epsilon}}$.

3.3 Solving the puzzle

In what follows, I display the results of the article.

3.3.1 Uniqueness

Using Section 3.2, I find the following.

Proposition 3 *If the following condition is verified:*

$$\left[\frac{\epsilon\mathbb{A}_1}{\gamma(\gamma+1)\phi}\right]^{\frac{1}{\gamma-\epsilon}} < \left(\frac{R_A}{\gamma\phi}\right)^{\frac{1}{\gamma+1}} \tag{11}$$

then a spatial equilibrium is unique.[3]

The first striking result is that a spatial equilibrium can be unique. This contradicts conventional wisdom. It is well acknowledged that there is a continuum of Pareto-ranked equilibria (see Henderson (1974), Behrens and Robert-Nicoud (2015) and Albouy et al. (2019)). The common belief is that the presence of agglomeration and dispersion forces makes the indirect utility non-monotonic, a feature that generates multiplicity. In the

[3]If condition (11) is verified, a spatial equilibrium is also stable. The proof is available upon request.

present article, I qualify this statement by showing that this depends on the value of agricultural rents.

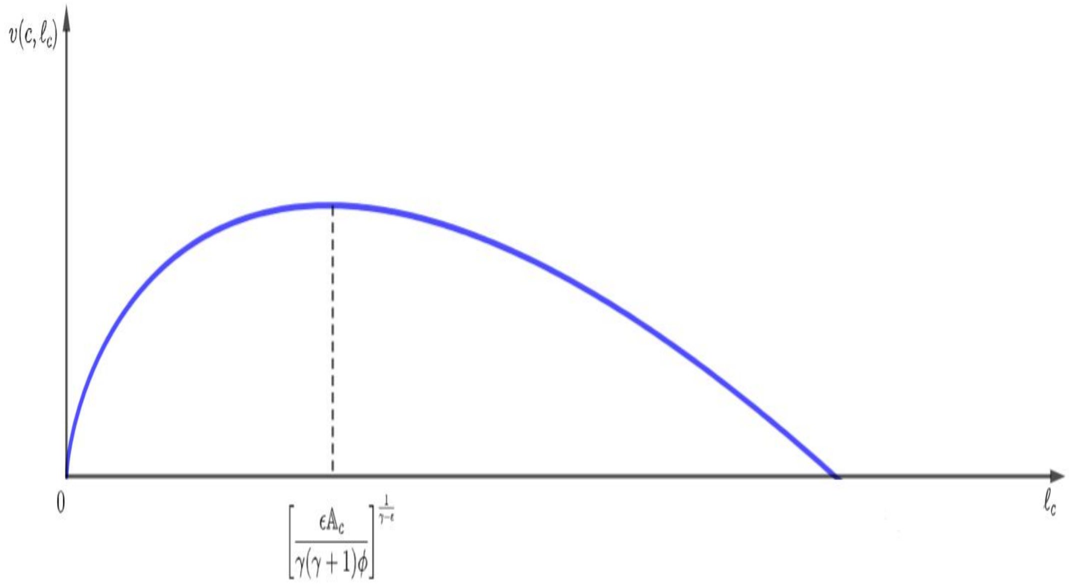

Figure 2: $R_A = 0$

The cases $R_A = 0$ and $R_A =$"small" nest the literature case. When agricultural rents are null/small, the utility level exhibits an inverted U-shape curve (see Figures 2 and 3). For small city size, agglomeration forces dominate dispersion forces, and v is monotonically increasing with population. Then, urban costs outweigh agglomeration economies beyond some "efficient" city size, and the indirect utility function becomes monotonically decreasing with population.[4] This creates multiplicity.

When agricultural rents are high enough, the indirect utility function is no longer a single-peaked function. The value of agricultural rent imposes for a lower bound for cities size, and this lower bound requires that cities size not lie to the rising part of the utility. This implies that the indirect utility function always decreases with respect to ℓ over its domain/interval (see Figures 4 and 5). This ensures that a unique spatial equilibrium exists.

Condition (11) is credible for two reasons. First, condition (11) is flexible and not demanding. This is because it can be complied by a large set of parameters. Many different calibrations can be considered such that a spatial equilibrium is unique.[5] Second, note that

[4]Efficient city size occurs when agglomeration economies offset congestion costs.

[5]It is possible to find a large range of different values for $(R_A, \phi, \gamma, \epsilon, \mathbb{A}_1)$ that satisfies equation (11).

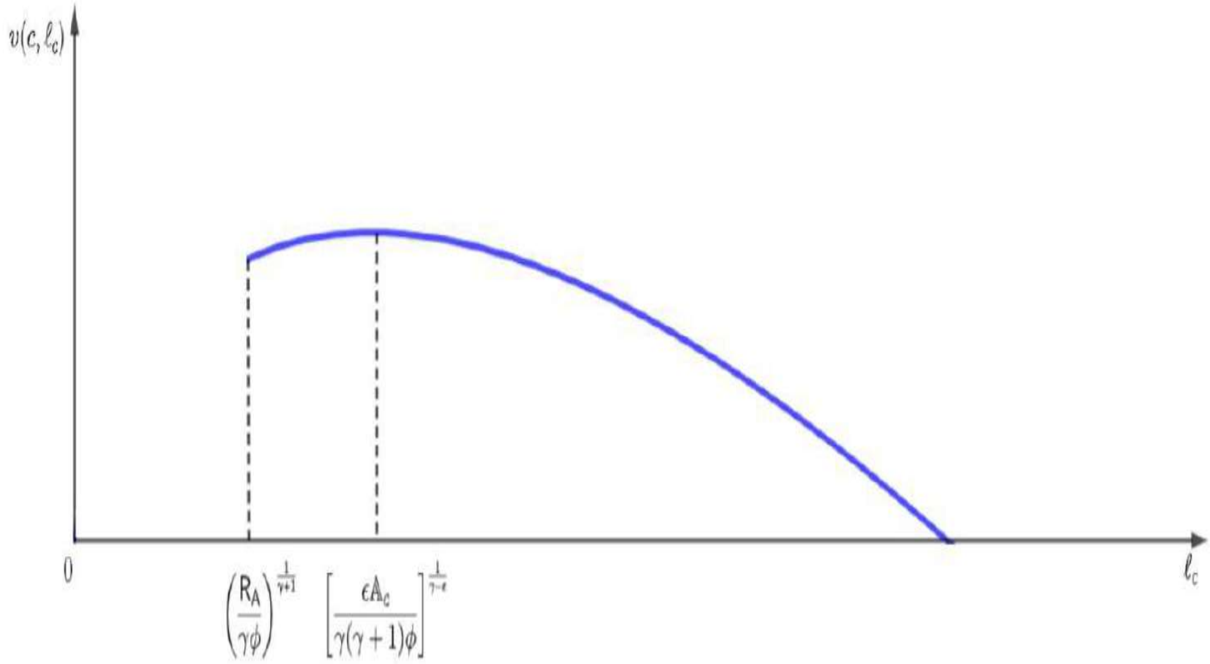

Figure 3: R_A small

if $\epsilon - \gamma \approx 0$ and if γ is small then condition (11) collapses to:

$$\left(\frac{\mathbb{A}_1}{\approx \phi}\right)^{\frac{1}{\approx 0}} < \left(\frac{R_A}{\approx \phi}\right)^{\approx 1} \tag{12}$$

The assumptions $\epsilon - \gamma \approx 0$ and γ being small find empirical support. Combes and Gobillon (2015) document that $\gamma - \epsilon$ is positive and small. Behrens and Robert-Nicoud (2015) report that the estimates of $\epsilon = 0.081$ and $\gamma = 0.088$, and find that the difference $\gamma - \epsilon$ is statistically indistinguishable from zero. Under this plausible restriction, condition (11) holds if the sufficient assumption is observed:

$$\mathbb{A}_1 < \phi \tag{13}$$

But, ϕ is a "free" parameter. As underlined by Boitier (2018), one does not have information about ϕ and the value of ϕ is inconsequential. In particular, because knowledge about ϕ is limited, Boitier (2018) lets this parameter adjust such that there exists a unique market equilibrium. Boitier (2018) also performs a sensitivity analysis and shows that varying the value of ϕ has no effects. The is because ϕ is a mere scaling factor in equation (3). As the utility function can be negative ($u < 0$), ϕ can have an arbitrary large value. As a consequence, it is possible to set ϕ large enough (see equation (13)) to ensure uniqueness without any difficulty. This suggests that having uniqueness is very common in the new setting.

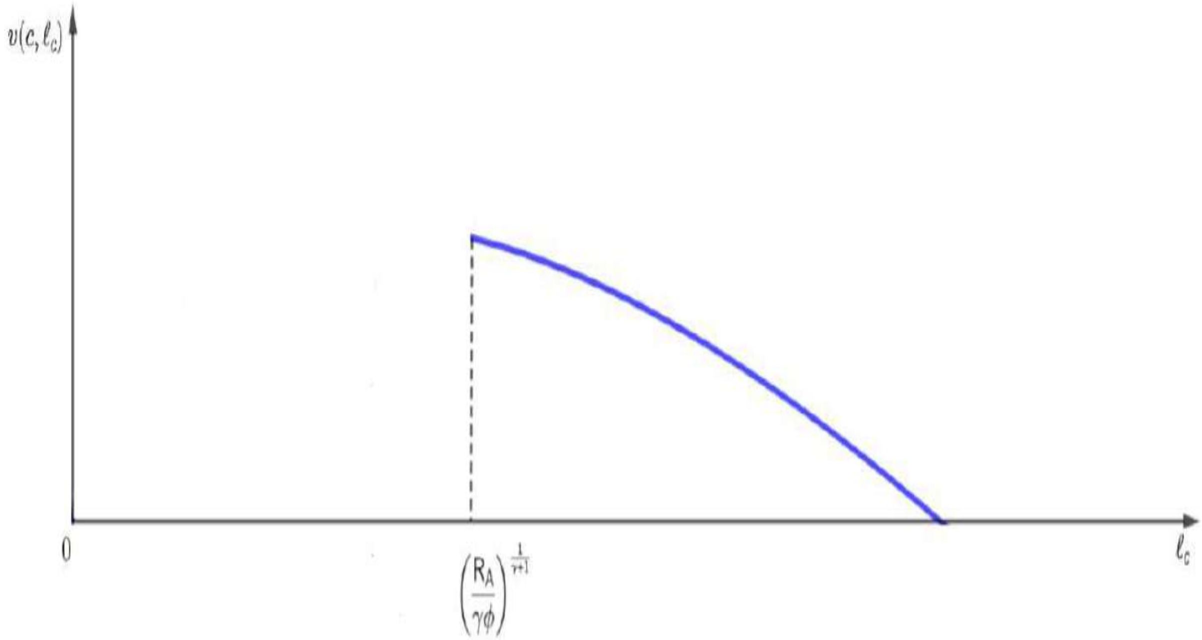

Figure 4: R_A high

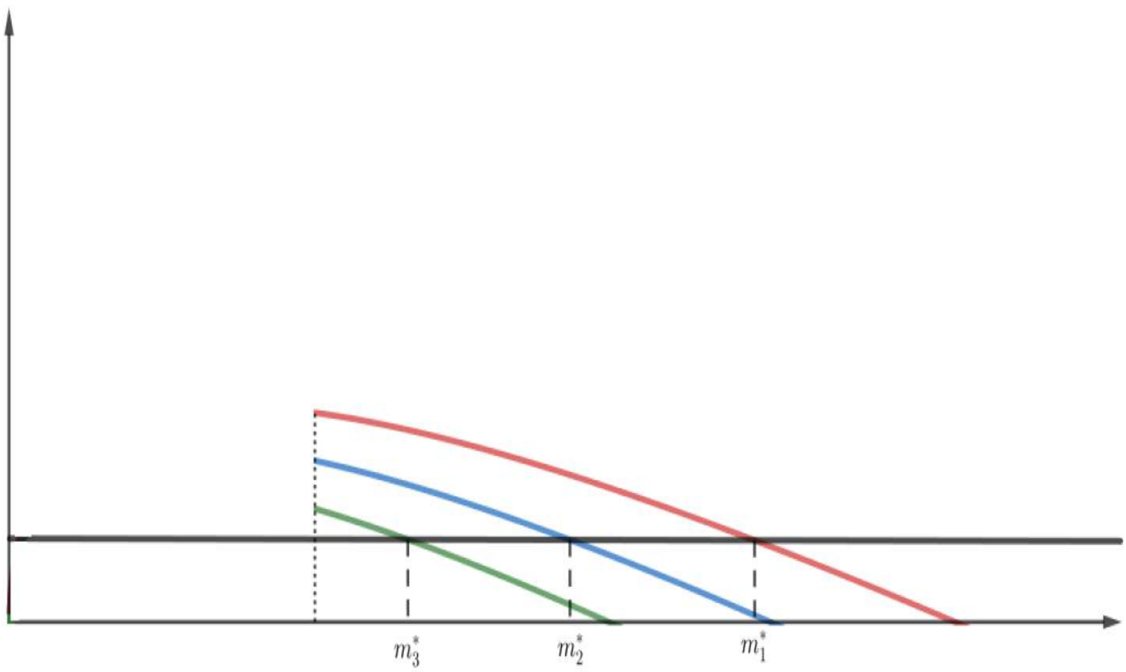

Figure 5: R_A high, three cities $(m = \ell)$

Remind that uniqueness is very desirable as it eliminates a formidable issue: the so-called "equilibrium selection problem". In standard systems of cities à la Henderson, the selection of an equilibrium is impossible. Another shortcoming lies in the great diversity of the features of equilibria. Equipped with (11), the present article fixes these issues. In sum, to obtain uniqueness in a system of cities à la Henderson, two ingredients are required:

i)- hyperbolic preferences to generate equation (1)

ii)- a simple condition on agricultural rents to ensure that the indirect utility function is monotonic over its domain (see equation (11))

3.3.2 Zipf's law

In this section, I show that the spatial equilibrium can follow a Zipf's law. Because the number of cities is fixed, the suitable definition for the Zipf's law is the following discrete version.

Definition 4 *ℓ^* is said to be a Zipf's law if the following is verified:*

$$\ell^* = (\ell_1^*, ..., \ell_C^*) \to Zipf(\varphi): \quad \ell_c^* = \frac{c^{-\varphi}}{H}$$

where $\varphi > 0$ and $H = \sum_{k=1}^{C} k^{-\varphi}$.

If the spatial distribution follows a Zipf's law then the rest of the spatial equilibrium is immediate and characterized by:

$$R_c^* = (\gamma\phi)\left(\frac{c^{-\varphi}}{\sum_{k=1}^{C} k^{-\varphi}}\right)^{\gamma+1}$$

and

$$z_c^* = \mathbb{A}_c\left(\frac{c^{-\varphi}}{\sum_{k=1}^{C} k^{-\varphi}}\right)^{\epsilon} - \gamma\phi\left(\frac{c^{-\varphi}}{\sum_{k=1}^{C} k^{-\varphi}}\right)^{\gamma}$$

for all $c \in \mathcal{C}$. Moreover, if equation (11) is satisfied then the spatial equilibrium is unique. As a consequence, a robust comparative satics analysis is available. Notably, the effect of ϕ is unambiguous. A increase in preference for land increases land prices but decreases consumption. This is intuitive. Households want to live in bigger houses and so bid for more units of land. As the endowment of land is fixed to unity in each city (see equation (7)), prices adjust and increase. This lowers the level of income dedicated to consumption, and equilibrium consumption is disminished.

Using Sections 3.2 and 3.3.1, I obtained the following.

Proposition 4 *Assume that (11) is satisfied. In addition, if the following condition is complied:*

$$\mathbb{A}_c = \mathbb{A}_1 c^{\varphi\epsilon} - (\gamma+1)\phi H^{\epsilon-\gamma}\left[c^{\varphi\epsilon} - c^{\varphi(\epsilon-\gamma)}\right] \tag{14}$$

then the spatial equilibrium is FS and follows the Zipf's law of Definition 4.

The second striking result of the present article is that a spatial equilibrium can follow a Zipf's law, depending on the value of TFP parameters. With homogeneous workers and a finite number of cities, Behrens and Robert-Nicoud (2015) state that the upper tail of the equilibrium city size distribution can inherit the properties of the distribution of $\mathbb{A}$ in the limit only. But, I highlight that this result is not motivated as the main condition behind this result rarely occurs (see Section 2.2). Here, condition (14) is more reasonable. Contrary to (2), condition (14) is compatible with the definition of a spatial equilibrium.

What is also appealing is that the shape of the distribution of $\mathbb{A}$ does not coincide with the shape of ℓ^*. To see that, consider the following peculiar case. Assume that $\epsilon - \gamma \approx 0$ such that condition (14) becomes the following:

$$\mathbb{A}_c \approx [\mathbb{A}_1 - (\gamma + 1)\phi]\, c^{\varphi\epsilon} + (\gamma + 1)\phi$$

If the following additional "knife edge" condition $\varphi\epsilon = 1$ is satisfied then the condition collapses to:

$$\mathbb{A}_c \approx [\mathbb{A}_1 - (\gamma + 1)\phi]\, c + (\gamma + 1)\phi$$

$\mathbb{A}$ must be affine to generate a Zipf's law. In other words, a linear distribution of $\mathbb{A}$ can generate the shape of the Zipf's law which is (by definition) very skewed. This indicates that the model endogenously generates a Zipf's law, which is a nice feature. The result is not *ad hoc* as it is not tied by the shape of $\mathbb{A}$.

3.4 Extension

So far, the preferences of workers are hyperbolic. These preferences are used because they have the ability to generate an indirect utility function of the form (1). However, they also are quasi-linear preferences implying that the income/revenue effect is eliminated in land consumption (see Proposition 1). As a consequence, the natural exercise to carry out is to gauge the capacity of the model to cope with preferences that include an income effect in land consumption.

Hereafter, I consider the previous model but with Cobb-Douglas preferences such that:

$$u(z_c, h_c) = z_c^{1-\gamma} h_c^{\gamma}, \quad 0 < \gamma < 1$$

where γ measures the role of housing in preferences. Adapting Propositions 1-4, I obtain the following proposition.

Proposition 5 *A spatial equilibrium is always FL. Then assume the following:*

$$\gamma - \epsilon(1 - \gamma) > 0 \tag{15}$$

then a spatial equilibrium is unique. In addition, if the following is verified:

$$\mathbb{A}_c = c^{-\varphi \times \frac{\gamma - \epsilon(1-\gamma)}{1-\gamma}} \tag{16}$$

then a spatial equilibrium follows the Zipf's law defined in Definition 4.

Three comments can be done.

First, a spatial equilibrium is always FL. This is explained by the form of the indirect utility function. In particular, v is unbouded by below: $\lim_{\ell \to 0} v(., \ell) = \infty$. This means that $\ell_c^* = 0$ for a fixed $c \in \mathcal{C}$ is untenable, and so $\ell_c^* > 0$ for all c in $\mathcal{C}$ is satisfied in equilibrium.

Second, a spatial equilibrium can be unique. This is because the indirect utility function is given by:

$$v(c, \ell_c) = \frac{(1-\gamma)^{1-\gamma} \mathbb{A}_c^{1-\gamma}}{\ell_c^{\gamma - \epsilon(1-\gamma)}}$$

If condition (15) is verified then v is always a decreasing function with respect to ℓ. This yields uniqueness.

Last, a spatial equilibrium can follows a Zipf's law if and only if the distribution of A follows a Zipf's law. The previous finding is therefore canceled out under Cobb-Douglas preferences.

4 Conclusions

It is well acknowledged that the standard systems of cities suffers from two shortcomings: a spatial equilibrium is not unique and cannot reproduce the basic fact that big cities follow an exact Zipf's law under credible conditions. In this article, I build a new theory of systems of cities that generates three new aspects. First, the standard indirect utility function à la Henderson is obtained using hyperbolic preferences. Second, a Zipf's law holds in the model, depending on the value of TFP parameters. Third, a sufficient condition on agricultural rents is available to ensure that a spatial equilibrium is unique. Having a unique spatial equilibrium in Henderson economies opens a new perspective for future research. Owing to uniqueness, the framework becomes amenable. It becomes possible to carry out robust comparative statics exercises, to measure welfare gains/losses, etc.

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A Proof of Proposition 1

Fix $c \in \mathcal{C}$. The following program:

$$\max_{h_c} \left\{ \mathbb{A}_c \ell_c^{\epsilon} - R_c h_c - \frac{\phi}{h_c^{\gamma}} \right\}$$

is well-posed such that h_c^* is determined by the following FOC:

$$-R_c - \frac{\gamma\phi}{(h_c^*)^{\gamma+1}} = 0$$

This leads to:

$$h_c^* = \left(\frac{\gamma\phi}{R_c}\right)^{\frac{1}{\gamma+1}}$$

Integrating this result into the expression z_c gives:

$$z_c^* = \mathbb{A}_c \ell_c^{\epsilon} - (\gamma\phi)^{\frac{1}{\gamma+1}} R_c^{\frac{\gamma}{\gamma+1}}$$

B Proof of Proposition 2

Noting that $h_c^* = \frac{1}{\ell_c^*}$ from equation (7) and plugging this relationship into equation (4) yields:

$$R_c^* = (\gamma\phi)(\ell_c^*)^{\gamma+1}$$

Integrating this result into z_c^* gives:

$$z_c^* = \mathbb{A}_c \ell_c^{*\epsilon} - (\gamma\phi)\ell_c^{*\gamma}$$

As $z_c^* > 0$ must be verified in equilibrium for all c in $\mathcal{C}$, this implies:

$$z_c^* > 0 \Leftrightarrow \ell_c^* < \left(\frac{\mathbb{A}_c}{\gamma\phi}\right)^{\frac{1}{\gamma-\epsilon}}$$

which imposes an upper-bound on ℓ^*. Similarly, using h_c^* and R_c^* equation (6) becomes:

$$v^* = \mathbb{A}_c \ell_c^{*\epsilon} - (\gamma+1)\phi\ell_c^{*\gamma} \quad \forall c \in \mathcal{C}$$

Integrating this result into Ψ, and as condition (5) holds for all c in $\mathcal{C}$, the following is verified:

$$\Psi(c, v^*) \geq R_A \Leftrightarrow \left(\frac{R_A}{\gamma\phi}\right)^{\frac{1}{\gamma+1}} \leq \ell_c^*$$

which imposes a lower-bound on ℓ^*.

C Proof of Proposition 3

The indirect utility function of the workers living in city $c \in \mathcal{C}$ is:

$$v(c, \ell_c) = \mathbb{A}_c \ell_c^{\epsilon} - (\gamma + 1)\phi \ell_c^{\gamma}$$

It is well known that a spatial equilibrium is unique if $\frac{\partial v(c,\ell_c)}{\partial \ell_c} < 0$ for all c in $\mathcal{C}$. Computing $\frac{\partial v(c,\ell_c)}{\partial \ell_c} < 0$ gives:

$$\left[\frac{\epsilon \mathbb{A}_c}{\gamma(\gamma+1)\phi}\right]^{\frac{1}{\gamma-\epsilon}} < \ell_c \tag{17}$$

as $\gamma > \epsilon$. Then remind two elements. First, ℓ_c is bounded by the following:

$$\left(\frac{R_A}{\gamma\phi}\right)^{\frac{1}{\gamma+1}} \leq \ell_c < \left[\frac{\mathbb{A}_c}{\gamma\phi}\right]^{\frac{1}{\gamma-\epsilon}}$$

with $\left[\frac{\epsilon \mathbb{A}_c}{\gamma(\gamma+1)\phi}\right]^{\frac{1}{\gamma-\epsilon}} < \left[\frac{\mathbb{A}_c}{\gamma\phi}\right]^{\frac{1}{\gamma-\epsilon}}$ as $\gamma > \epsilon$. Second, the vector $\mathbb{A}$ is ordered such that: $\mathbb{A}_1 > ... > \mathbb{A}_C$. As a consequence, if the following is verifies

$$\left[\frac{\epsilon \mathbb{A}_1}{\gamma(\gamma+1)\phi}\right]^{\frac{1}{\gamma-\epsilon}} < \left(\frac{R_A}{\gamma\phi}\right)^{\frac{1}{\gamma+1}}$$

then (17) is satisfied for all c in $\mathcal{C}$. This implies uniqueness.

D Proof of Proposition 4

Assume that condition (11) holds. Also assume that the spatial equilibrium follows the following Zipf law:

$$\ell^* = (\ell_1^*, ..., \ell_C^*) \rightarrow Zipf(\varphi): \quad \ell_c^* = \frac{c^{-\varphi}}{H}$$

where $\varphi > 0$ and $H = \sum_{c=1}^{C} c^{-\varphi}$. Fix a city $c \in \mathcal{C}$. Then note that the following must be verified:

$$\mathbb{A}_1 \ell_1^{*\epsilon} - (\gamma+1)\phi \ell_1^{*\gamma} = \mathbb{A}_c \ell_c^{*\epsilon} - (\gamma+1)\phi \ell_c^{*\gamma}$$

that is

$$\mathbb{A}_c = \mathbb{A}_1 c^{\varphi\epsilon} - (\gamma+1)\phi H^{\epsilon-\gamma}\left[c^{\varphi\epsilon} - c^{\varphi(\epsilon-\gamma)}\right]$$

using the expressions of ℓ_1^* and ℓ_c^*. In sum, if the above condition is verified then the spatial equilibrium follows the Zipf law with parameter φ.

E Proof of Proposition 5

Fix $c \in \mathcal{C}$. The following program:

$$\max_{h_c} \left\{ (\mathbb{A}_c \ell_c^\epsilon - R_c h_c)^{1-\gamma} h_c^\gamma \right\}$$

is well-posed such that h_c^* is determined by the following FOC:

$$-(1-\gamma) R_c (\mathbb{A}_c \ell_c^\epsilon - R_c h_c^*)^{-\gamma} h_c^{*\gamma} + \gamma (\mathbb{A}_c \ell_c^\epsilon - R_c h_c^*)^{1-\gamma} h_c^{*\gamma-1} = 0$$

This leads to:

$$h_c^* = \frac{\gamma \mathbb{A}_c \ell_c^\epsilon}{R_c}$$

Integrating this result into the expression z_c gives:

$$z_c^* = (1-\gamma)\mathbb{A}_c \ell_c^\epsilon \geq 0 \quad \forall c \in \mathcal{C}$$

and the indirect utility function becomes:

$$v(c, \ell_c) = \frac{(1-\gamma)^{1-\gamma} \mathbb{A}_c^{1-\gamma}}{\ell_c^{\gamma - \epsilon(1-\gamma)}}$$

If the following condition is verified:

$$\gamma - \epsilon(1-\gamma) > 0$$

then $\frac{\partial v(.,\ell)}{\partial \ell} < 0$ holds and a spatial equilibrium is unique. In addition, as $\lim_{\ell \to 0} v(., \ell) = +\infty$ is verified a spatial equilibrium is always full support and can be explicitly pinned down. To see that, note that the following is satisfied in equilibrium:

$$\ell_c^* = \left[\frac{(1-\gamma)^{1-\gamma} \mathbb{A}_c^{1-\gamma}}{v^*} \right]^{\frac{1}{\gamma - \epsilon(1-\gamma)}}$$

for all c in $\mathcal{C}$.Then I note that:

$$\sum_{c=1}^{C} \ell_c^* = 1 \Leftrightarrow \left[\frac{(1-\gamma)^{1-\gamma}}{v^*} \right]^{\frac{1}{\gamma - \epsilon(1-\gamma)}} = \frac{1}{\sum_{c=1}^{C} \mathbb{A}_c^{\frac{1-\gamma}{\gamma - \epsilon(1-\gamma)}}}$$

This implies that:

$$\ell_c^* = \frac{\mathbb{A}_c^{\frac{1-\gamma}{\gamma - \epsilon(1-\gamma)}}}{\sum_{c=1}^{C} \mathbb{A}_c^{\frac{1-\gamma}{\gamma - \epsilon(1-\gamma)}}}$$

As a consequence, if the following condition:

$$\mathbb{A}_c = c^{-\varphi \times \frac{\gamma - \epsilon(1-\gamma)}{1-\gamma}}$$

is satisfied for all c in $\mathcal{C}$ then the unique spatial equilibrium follows a Zipf's law.