Introduction

The converted waves have gained more attention due to the creation of converted-wave image or PS image with the developments in multicomponent surveying. The subject of converted waves is receiving a new attention, principally because of the new practicality of multicomponent ocean-bottom seismology (OBS) and the high data quality often achieved. Compared with the pure wave-mode processing, the converted-wave processing is more dependent on physical assumptions like the rock velocities because both moveout and the offset of the imaging point depend on the physical parameters of the medium. The converted-wave traveltimes is an important issue in converted-waves data processing such as velocity analysis, modeling, and time migration. In the VTI model, there is only one converted wave (P-S conversion). The traveltimes for the converted PS wave could be computed by summation of P and S waves traveltimes. Several papers dealing with the converted-wave traveltimes in VTI media have been published in the past (Thomsen 1999; Stewart et al. 2003). The orthorhombic (ORT) model was investigated and has gained more attention due to the need to characterize the fractured earth. The kinematic parameters of pure- and converted-mode waves in elastic ORT media are derived by Stovas (2017). Compared with the VTI model, the study of the converted traveltimes in the ORT model is much more complicated since there are more types of converted waves and due to the existence of the singularity point (Ivanov and Stovas 2017). No explicit expression for the converted wave in elastic ORT is derived so far and this blank is still waiting to be filled.

In order to derive an explicit expression of the converted-wave traveltimes, we define the rational-form approximation for the converted PS1, PS2 and S1S2 waves in an elastic ORT model. In order to obtain the approximation coefficients, the Taylor-series approximation in the corresponding vertical slowness for three pure-wave modes is applied. The coefficients in the approximation could be expressed by these series coefficients and further shown by the anisotropy parameters. The error in the traveltimes of the converted PS1, PS2 and S1S2 waves are tested for three defined ORT models in the numerical examples. We find that for PS1, PS2 waves, the results of the approximation are very accurate for all the models we tested. For S1S2 wave, the approximation is only accurate at near offsets due to the existence of the singularity point.

Rational form traveltimes approximation for the converted wave in a VTI model

The behavior of a converted PS wave in anisotropic media is rather complex compared with the isotropic case. The sketch of a converted PS wave in a homogeneous VTI model is shown in Figure 1 (left). In that case, the conversion-point position is given by the downward-propagating P wave offset. The offset of the converted wave is computed by summing the corresponding P and S wave offsets,

\[ x_C = x_p + x_s, \quad t_C = t_p + t_s. \]  

The parameters \( x_p \) and \( x_s \) are the offsets for P and S wave, respectively, and \( t_p \) and \( t_s \) are the one-way traveltimes for P and S wave, respectively. The offset for P and S waves is computed from the derivative of the corresponding vertical slowness,

\[ x_i = -z_0 dp_{z,i} / dp, \quad t_i = z_0 p_{z,i} + x_i p_z, \]

where \( i=P, S \) represents the corresponding wave mode and \( z_0 \) is the depth of the reflector, \( p_z \), and \( p_s \) are the projections along the horizontal and vertical direction, respectively.

The rational-type (Thomsen 1999) approximation in traveltimes for the converted wave is defined by

\[ t_C^2 = a_0 + a_2 x_C^2 + a_4 x_C^4 / (1 + b_2 x_C^2), \]

The coefficients \( a_0, a_2, \) and \( a_4 \) are computed from the zero-offset limit, \( b_2 \) is computed from the infinite-offset limit \( \lim t_C^2 / x_C^2 = k \) and given by \( b_2 = a_2 / (k - a_4) \).

Form of the coefficients \( a_0, a_2, \) and \( a_4 \) in the rational-form approximation (3) are given by

\[ a_0 = \left( V_p^2 + V_S^2 \right) v_0^2, \quad a_2 = \left( V_p^2 + V_S^2 \right) \left[ V_p^2 (V_p^2 + V_S^2 (1 + 2 \delta_i) + 2 V_p^2 (\delta_i - \delta_s)) \right], \]

\[ a_4 = -4V_p^2 v_0^2 \left( V_p^2 + V_S^2 (1 + 2 \delta_i) + 2 V_p^2 (\delta_i - \delta_s) \right)^2. \]
The parameters $V_{p0}$ and $V_{s0}$ are the vertical velocities for P and S wave, respectively, $\varepsilon_1$ and $\delta_1$ are the anisotropy parameters defined in Thomsen (1986).

In order to test the accuracy of the rational-form approximation, we introduce two VTI models with parameters defined in Table 1. The relative error in traveltime of the converted wave using two VTI models is shown in Figure 1 (right). One can tell from the plot that the approximation is very accurate for both VTI models.

![Figure 1](image)

**Figure 1.** (left) Conversion-point position schematic in a homogeneous VTI medium. (right) The relative error in traveltime for the converted PS wave in VTI model. The errors computed from VTI model 1 and 2 are plotted by the solid and dashed lines, respectively.

<table>
<thead>
<tr>
<th>Model</th>
<th>$V_{p0}$ (km/s)</th>
<th>$V_{s0}$ (km/s)</th>
<th>$\varepsilon_1$</th>
<th>$\delta_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Model 1</td>
<td>2</td>
<td>1</td>
<td>0.22</td>
<td>0.1</td>
</tr>
<tr>
<td>Model 2</td>
<td>2</td>
<td>1</td>
<td>0.35</td>
<td>0.2</td>
</tr>
</tbody>
</table>

**Table 1.** The model parameters for two VTI models.

### Rotational-form approximation of traveltime for converted waves in the elastic ORT medium

For a pure wave mode ($i=S_1, S_2$ or $P$), the corresponding offset is obtained by the corresponding derivatives:

$$x_i = -z_i \frac{\partial p_{x,i}}{\partial x_i}, \quad y_i = -z_i \frac{\partial p_{y,i}}{\partial y_i}, \quad t_i = z_i p_{x,i} + y_i p_{y,i}. \quad (4)$$

For a converted wave, the offsets and traveltime are computed by

$$x_j = x_i + x_j, \quad y_j = y_i + y_j, \quad t_j = t_i + t_j, \quad (5)$$

where $i, j = P, S_1, S_2$ indicate the corresponding combination of the converted waves. The diagram for the converted PS wave in a homogeneous ORT model is taken as an example and shown in Figure 2. The offsets and traveltime for the converted PS wave are computed by summing the offsets and traveltimes of the P and S waves.

The rational-form traveltime for the converted wave is defined by

$$t_{ij}^2 = M_{ij0} + M_{20,ij} x_i^2 + M_{02,ij} y_i^2 + \left(M_{40,ij} x_i^4 + M_{22,ij} x_i^2 y_i^2 + M_{04,ij} y_i^4\right) / \left(1 + N_{20,ij} x_i^2 + N_{02,ij} y_i^2\right). \quad (6)$$

The coefficients $M_{0,ij}, M_{20,ij}, M_{40,ij}, M_{02,ij}, M_{04,ij}$ and $M_{22,ij}$ are computed from the zero-offset limits and the coefficients $N_{20,ij}$ and $N_{02,ij}$ are computed from the infinite-offset limits

$$\lim_{x_i \to 0} t_{ij}^2 / x_i^2 = K_x, \quad \lim_{y_i \to 0} t_{ij}^2 / y_i^2 = K_y, \quad (7)$$

with the forms given by

$$N_{20} = M_{40} / (K_x - M_{20}), \quad N_{02} = M_{04} / (K_y - M_{02}). \quad (8)$$

In order to obtain these coefficients, the trail solution for the corresponding vertical slowness squared for the three pure wave modes given by the effective model parameters (Stovas 2017) is applied:

$$p_{z,i}^2 = A_{0,i} + A_{20,i} p_z^2 + A_{02,i} p_y^2 + A_{40,i} p_i^4 + A_{04,i} p_z^4 + A_{22,i} p_y^2 p_z^2, \quad (9)$$

where $i=P, S_1, S_2$ indicates the different pure wave modes. The coefficients $M_{0,ij}, M_{20,ij}, M_{40,ij}, M_{02,ij}, M_{04,ij}$, and $M_{22,ij}$ in the rational-form approximation are expressed through the Taylor series: $A_{0,i}, A_{20,i}, A_{40,i}, A_{02,i}, A_{04,i}$, and $A_{22,i}$ computed for the pure wave modes. These coefficients are shown by the effective model parameters: $V_{0,i}, V_{SMO1,i}, V_{SMO2,i}, \eta_{1,i}$, and $\eta_{2,i}$ for P, $S_1$ and $S_2$ waves in Xu and Stovas (2019).
Figure 2. Diagram for the converted PS\(_1\) wave in a homogeneous ORT model. The offsets and traveltimes for the converted PS\(_1\) wave are computed by summing the offsets and traveltimes of the P and S\(_1\) waves with \(x_{\text{PS}1} = x_P + x_{S1}\), \(y_{\text{PS}1} = y_P + y_{S1}\), and \(t_{\text{PS}1} = t_P + t_{S1}\).

**Numerical examples**

In order to see the accuracy of the rational approximation presented by equation (6), we show the relative error for three converted waves: PS\(_1\), PS\(_2\), and S\(_1\)S\(_2\) by

\[
E_{ij} = (t_{ij}' - t_{ij}) \times 100/t_{ij}'
\]  

(10)

Note that \(t_{ij}'\) is the rational-form approximation for the converted wave and the exact traveltine \(t_{ij}\) is computed from the dynamic ray tracing. The relative error for three converted waves: PS\(_1\), PS\(_2\), and S\(_1\)S\(_2\) computed for three ORT models versus two offset-depth ratios are shown in Figure 3, 4, and 5, respectively. Note that the model parameters for the three ORT models are defined in Table 2. One can see from the plots that for ORT model 1, the rational-form approximations for the converted PS\(_1\) and PS\(_2\) waves are very accurate while for the converted S\(_1\)S\(_2\) wave the accuracy is relatively low. The maximal errors for all converted waves are obtained along 45-degree azimuth. For the numerical examples for ORT model 2, the errors are all very small for all converted waves. The error for the converted PS\(_1\) wave is increasing along the Y-axis while for the converted PS\(_2\) wave, it is along the X-axis. The shape of the error surface for S\(_1\)S\(_2\) wave is more complicated. Similar to the case for ORT model 2, the traveltine errors computed for ORT model 3 are all very small for the three converted waves. For the converted PS\(_1\) and PS\(_2\) waves, the maximal errors are obtained along the 45-degree azimuth while for the converted S\(_1\)S\(_2\) wave, the error is increasing along the X-axis.

Note that for the converted PS\(_1\) and PS\(_2\) waves, the compensation from P wave can overcome the singularity effect from S-waves, therefore, there is no singularity for converted PS\(_1\) and PS\(_2\) waves along the symmetry planes (Roganov and Stovas 2010). For a P-wave, the sign is always negative, and the magnitude is relatively lager than those of the S-waves, which compensates some of the triplicated area caused by S waves for the converted PS waves. For S\(_1\) and S\(_2\) and converted S\(_1\)S\(_2\) waves, the singularity could exist depending on the model parameters. For the results of the tested ORT models 2 and 3, the error in traveltine for converted S\(_1\)S\(_2\) wave is also very small but the accuracy could only be preserved at small offsets. When we increase the offsets for the computation, the error for the converted S\(_1\)S\(_2\) wave will become extremely large because of the singularity effect. Moreover, due to the existence of cusps, triplications, and shear singularities, the converted S\(_1\)S\(_2\) wave signal is very unstable and, in practical application, S\(_1\)S\(_2\) wave data is seldomly used in field applications.

<table>
<thead>
<tr>
<th></th>
<th>(V_{P0}(\text{km/s}))</th>
<th>(V_{S0}(\text{km/s}))</th>
<th>(\varepsilon_1)</th>
<th>(\delta_1)</th>
<th>(\gamma_1)</th>
<th>(\varepsilon_2)</th>
<th>(\delta_2)</th>
<th>(\gamma_2)</th>
<th>(\delta_3)</th>
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<tbody>
<tr>
<td>Model 1</td>
<td>3.332</td>
<td>1.732</td>
<td>0.216</td>
<td>0.169</td>
<td>0.059</td>
<td>0.198</td>
<td>0.274</td>
<td>0.133</td>
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</tr>
<tr>
<td>Model 2</td>
<td>3.5</td>
<td>1.53</td>
<td>-0.145</td>
<td>-0.178</td>
<td>-0.106</td>
<td>0.041</td>
<td>-0.102</td>
<td>-0.013</td>
<td>-0.065</td>
</tr>
<tr>
<td>Model 3</td>
<td>4.625</td>
<td>2.751</td>
<td>-0.179</td>
<td>-0.142</td>
<td>-0.068</td>
<td>-0.068</td>
<td>-0.097</td>
<td>-0.013</td>
<td>0.303</td>
</tr>
</tbody>
</table>

Table 2. The model parameters for three ORT models.

**Conclusions**

We define the rational-form approximation in traveltine for the converted PS\(_1\), PS\(_2\), and S\(_1\)S\(_2\) waves in the elastic orthorhombic (ORT) medium. The Taylor-series in corresponding vertical slowness is applied to the three pure wave modes to compute the approximation coefficients. Represented by the anisotropy parameters in the elastic ORT model, the explicit-form expression of the converted-wave traveltine is obtained. Using numerical modeling, we tested our proposed approximation for three ORT models. The results of the converted-wave traveltine are accurate for the converted PS\(_1\) and PS\(_2\) waves for all tested ORT models. For S\(_1\)S\(_2\) wave, due to the existence of the singularity point, the result is only accurate at near offsets.
References

**Figure 3.** The relative error in traveltime for the converted waves computed for ORT model 1. The errors computed for the converted PS$_1$, PS$_2$, and S$_1$S$_2$ waves are shown in (a-c), respectively. The model parameters for ORT model 1 are defined in Table 2.

**Figure 4.** The relative error in traveltime for the converted waves computed for ORT model 2. The errors computed for the converted PS$_1$, PS$_2$, and S$_1$S$_2$ waves are shown in (a-c), respectively. The model parameters for ORT model 2 are defined in Table 2.

**Figure 5.** The relative error in traveltime for the converted waves computed for ORT model 3. The errors computed for the converted PS$_1$, PS$_2$, and S$_1$S$_2$ waves are shown in (a-c), respectively. The model parameters for ORT model 3 are defined in Table 2.