Introduction

Seismic forward modeling is the engine of seismic imaging algorithms and the integral part of seismic data processing and survey design. Based on the medium and purpose of simulation, different forms of the wave equation can be considered and solved. In this regard, different families of the numerical approaches have been proposed, such as Finite Difference method (FD) (Alford et al., 1974), Finite Volume Method (FVM) (Dumbser et al., 2007), Finite Element Method (FEM) (Lysmer and Drake, 1972), Spectral Element Method (SEM) (Seriani and Priolo, 1994), and Discontinuous Galerkin method (DGM) (Riviere and Wheeler, 2003). Each method has its own advantages and drawbacks in terms of accuracy and computational aspects.

SEM initially developed for the fluid dynamics problems (Patera, 1984) and then, it has been applied to the wave equation successfully. This method provides higher accuracy than the FDM, also it is more geometrically flexible (De Basabe, 2009). The main characteristic of the SEM is use of high-order basis functions, which provides a lower sampling ratio for the wavefield and thereby reduces computational cost compared to the FEM. DGM was initially proposed to solve the hyperbolic neutron transport equation by Reed and Hill (1973). The DGM is a generalization of the FEM which allows the wavefield to be discontinuous at the element interfaces (De Basabe et al., 2016). The DGM is particularly suited to incorporate fractures and faults. However, its computational cost is higher than SEM.

In this study, the Interior Penalty Discontinuous Galerkin Method (IP-DGM) has been formulated and implemented for the acoustic wave propagation simulation. We investigate the accuracy of the IP-DGM and compare it with the SEM. The IP-DGM allows considering discontinuities in the model. This discontinuity can be attributed to the discontinuities of the elastic properties of the model such as fractures (Liu et al. 1995) as well as any other discontinuity related to the wavefield. In this study, we consider the cases where nodal basis functions have different orders inside the model and consequently the nodes on the adjacent elements with different order basis functions don’t fit together. SEM sees this as the discontinuity and reflections appear from those interfaces while IP-DGM can resolve this problem.

In the next section, we briefly describe the theory of the IP-DGM and then we show numerical results and finally we state our conclusions.

Theory

The DGM is a high-order finite element method which allows the approximating functions to be discontinuous at the element interfaces. By adding extra terms in the weak formulation, the continuity of the displacement field could be imposed. There are many different ways to impose, which leads to different forms of the DGM. We use the symmetric interior-penalty Galerkin method (SIPG) (De Basabe, 2009) in this study. The strong form of the time-domain acoustic wave equation in a homogeneous medium is

$$\partial_{tt} P - c^2 \nabla^2 P = f$$

To obtain a weak formulation suitable for DGM, multiplying the wave equation by $\delta P$, integrating element-wise, and adding the contributions from all the elements yields (De Basabe, 2009)

$$\sum_{E \in \Omega_h} ((c^{-2} \partial_{tt} P, \delta P)_E + B_E(P, \delta P)) + \sum_{\gamma \in \Gamma_h} J_\gamma(P, \delta P; S, R) = \sum_{E \in \Omega_h} (f, \delta P)_E$$

Where,

$$(u, v)_E = \int_E u v \, dx \, dz,$$

$$B_E(P, \delta P) = \int_E \nabla P \cdot \nabla \delta P \, dx \, dz,$$

$$J_\gamma(P, \delta P; S, R) = - \int_\gamma \{\nabla P \cdot n_\gamma\} [\delta P] \, dy + S \int_\gamma \{\nabla \delta P \cdot n_\gamma\} [P] \, dy + R \int_\gamma [P][\delta P] \, dy.$$
In the above equations, $P$ is the pressure wavefield, and $c$ is the propagation velocity. Furthermore, $\Omega$ represents the computational domain that can be 2D or 3D, $\Omega_h$ is finite element partition of $\Omega$, $\Gamma_h$ is set of faces between elements of $\Omega_h$, $\delta$ is the variation operator, and $n$ is the unit normal vector. $n_i$ is the number of nodes per element, $N_i$ is the interpolation function or element shape functions of the corresponding node, $p_i$ is the nodal values of the model response. Also, let $\gamma$ be the edge between two elements then the jump function denotes $[.]$ and $\{.\}$ the average function on the $\gamma$. The parameter $R$ is the penalty, and $S$ is a parameter that takes the values 0, 1 or -1 depending on the particular formulation of IP-DGM; we use $S = -1$ for SIPG. All the results in the following section were computed by using quadrilateral elements, and SIPG using nodal basis functions in two dimensions.

In the following, we consider a simple model with artificial discontinuity related to the change of order of nodal basis functions, where some nodes do not connect to each other on the boundary of the discontinuity. Using basis functions with different orders inside the model makes the modeling more flexible and can reduce the total number of the nodes and thereby the computational cost. For instance, to model the surface waves or low velocity body waves in a shallow part of the model (LVL layer of water layer) we require to consider dense grids at the surface, while grids spacing can be coarse inside the model. SEM doesn’t support the change of order of the basis functions and spurious reflections arise from those interfaces. In this study, we show how IP-DGM can accommodate this problem.

**Examples**

Here we provide the numerical results for a homogenous acoustic model. The model is a homogeneous model of 1 Km by 1 Km in $x$ and $z$ directions with velocity of 2.0 Km/s, the source is a Ricker wavelet with dominant frequency of 20 Hz located at $z = 500m$ and $x = 500m$ and the receivers are located at $z = 500m, x = 750m$ ($R_1$) and $z = 750m, x = 750m$ ($R_2$). To evaluate the numerical accuracy of the SEM and IP-DGM approaches (using the 5’th order of the Gauss-Lobatto-Legendre (GLL) nodes) we consider three element sizes: $20m$, $40m$, and $80m$ and compare numerical results with the analytic solution (Figure1).

![Figure 1](image1.png)
Figure 1 Comparison between SIPG, SEM, and analytic solution. Numerical dispersion is negligible for element size of 20 m (a and b) which corresponds to four samples per minimum wavelength. Considering larger element size (c, d, e, and f) causes numerical dispersion. This model is homogeneous, so the penalty term of IP-DGM vanishes between elements, therefore the results of the SEM and IP-DGM are identical.

To show the performance of the IP-DGM in resolving discontinuities, we consider a model with discontinuity in nodes due to the change of order of basis functions. The model is a homogeneous model of dimension 1 Km by 1 Km with a velocity of 2.0 Km/s; the source is the Ricker wavelet with a dominant frequency of 30 Hz located at \( z = 0 \), and \( x = 500 \) m, and the receiver located at \( z = 250 \) and \( x = 500 \). Figure 2 shows the structure of the elements where discontinuity exists in the middle of the model due to considering third and second order basis functions for the upper and lower part of the model respectively. The second-order finite difference method has been used for time-stepping.

![Figure 2](image)

Figure 2 left) shows a model with a discontinuity at the middle part, right) the geometrical location of the nodes along discontinuity. As can be seen, some black nodes don’t coincide with the orange nodes.

Figure 3 shows a comparison of the SEM and IP-DGM methods for the model shown in Figure 2. According to this figure, strong reflection happens from this discontinuity using SEM while setting an appropriate penalty parameter this spurious reflection can be attenuated effectively using IP-DGM. We have examined a wide range of the penalty parameter and finally found the penalty proposed by Ainsworth et al. (2006) can suppress reflections in an efficient way. According to our tests, \( R = 0.75 \) is the best value for the penalty parameter.
Conclusions

In this study, the accuracy of IP-DGM and SEM methods have been compared. The SIPG method with nodal-basis functions provides virtually equal accuracy to SEM. It should be noted that IP-DGM has the advantage over SEM that can detect mismatched finite element meshes. In a model with discontinuity in nodes due to the change of order in basis functions, SEM made a strong reflection. A wide range of the penalty parameter for IP-DGM have been examined, finally, the parameter which is proposed in the research conducted by Ainsworth and his co-workers in 2006, gives the best answer.

References

De Basabe, J. [2009]. High-order finite element methods for seismic wave propagation. Texas, University of Texas at Austin.