A new implementation of CPML for the second-order wave equation

Introduction

Finite difference (FD) methods have been widely used to numerically simulate wave equations for seismic wavefield simulation. Restricted by the limited resource of computers, artificial truncation to the model is inevitable, which generates spurious reflections from the artificial boundary. As such, boundary conditions should be used to mitigate the spurious reflections from the artificial boundary. The perfectly matched layer (PML) method, first proposed by Bérenger (1994) for the first-order Maxwell's equations, has proven to be an efficient absorbing boundary condition. Kuzuoglu and Mittra (1996) introduced a strictly causal form of the PML called the complex frequency-shifted PML (CFS PML), which has a better absorbing performance for low frequency waves and waves at grazing incidence. Roden and Gedney (2000) proposed the implementation of CFS PML based on a recursive convolution technique. Many studies extended the CPML method to the first-order seismic wave equation systems (Komatitsch and Martin, 2007).

The CPML implementation can apply to first-order equation systems easily but not to second-order equation systems. Many researchers have studied the PML boundary conditions for second-order equations system (Komatitsch and Tromp, 2003; Li and Matar, 2010; Ma et al., 2018). All of these CPML implementation methods for second-order equation systems need to artificially introduce auxiliary equations or auxiliary variables to rewrite the CPML control equation, which will lead to the increase of calculation, more auxiliary variables, and lower absorption efficiency.

In this paper, we present a new implementation of unsplit CPML for the second-order wave equation system. We divide the calculation area into several blocks, and use the corresponding FD scheme in each calculation area. In the PML domain, we propose to simulate second-order CPML equation based on two-step strategy (TS-CPML for short). This method can simulate the second-order CPML equation directly avoiding introducing some auxiliary variables and equations for rewriting it. Implementing TS-CPML boundary condition into existing FD simulation codes is straightforward and the same FD scheme can be used in both the physical model region and the PML region.

Method

The complex frequency-shift (CFS) stretching function \( \hat{s}_x \) can be expressed as:

\[
\hat{s}_x = \kappa_x + \frac{d_x}{a_x + i\omega},
\]

where \( d_x \) is the dumping function and can be expressed as \( d_x = d_0 (x/L)^2 \) in the PML domain, where \( d_0 \) can be expressed as \( d_0 = 3\nu \log((1+R_s)/(2L)) \), \( L \) is the thickness of the PML, \( x \) indicates the distance of the point in the PML to the PML interface along the \( x \) axis. \( a_x \geq 0 \) is the frequency-shifted factor and can be expressed as \( a_x = a_0 (1-x/L) \), where \( a_0 \) can be expressed as \( a_0 = 2\pi f_0 \). \( \kappa_x \geq 1 \) is the scaling factor, which can be expressed as \( \kappa_x = 1+(\kappa_0 -1)(x/L)^2 \), where \( \kappa_0 \) is a constant value, which can usually be set between 1 and 20.

Following Collino and Tsogka (2001), the relationship between the complex (stretched) coordinate \( \hat{x} \) in the PML domain and the original coordinate \( x \) can be written as:

\[
\frac{\partial}{\partial \hat{x}} = \frac{1}{\hat{s}_x} \frac{\partial}{\partial x}.
\]

In the time-domain, the partial derivative in the stretched coordinate can be written as a convolution between the inverse Fourier transform of the reciprocal of the stretch function and the regular partial derivative (Komatitsch and Martin, 2007; Pasalic and McGarry, 2010):
\[ \frac{\partial}{\partial x} = \frac{1}{\kappa_x} \frac{\partial}{\partial x} + \psi_x, \quad \text{where} \quad \psi_x = \frac{d}{\kappa_x} H(t) e^{-\frac{\Delta_x}{\kappa_x}} \frac{\partial}{\partial x} \]  

(3)

where \( \ast \) denotes the convolution operation, \( H(t) \) is the Heaviside function and \( \psi_x \) is known as the time convolution term and, in the discrete-time domain, it can be calculated in a recursive format:

\[ \psi_x^{(n)} = b_x \psi_x^{(n-1)} + c_x \frac{\partial}{\partial x}, \quad \text{where} \quad b_x = e^{-\frac{\Delta_x}{\kappa_x}}, \quad c_x = \frac{d}{\kappa_x} (b_x - 1) \]  

(4)

The original second-order elastic CPML equation in a homogenous isotropic model in terms of the displacements in Cartesian coordinates can be written as:

\[ \rho \frac{\partial^2 u}{\partial t^2} = (\lambda + 2\mu) \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} \right) + (\lambda + \mu) \frac{\partial}{\partial x} \left( \frac{\partial v}{\partial x} \right) + \mu \frac{\partial}{\partial x} \left( \frac{\partial w}{\partial x} \right) \]

\[ \rho \frac{\partial^2 w}{\partial t^2} = (\lambda + 2\mu) \frac{\partial}{\partial x} \left( \frac{\partial w}{\partial x} \right) + (\lambda + \mu) \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} \right) + \mu \frac{\partial}{\partial x} \left( \frac{\partial v}{\partial x} \right) \]  

(5)

where \( \lambda \) and \( \mu \) are the Lamé parameters. Obviously, by simplifying the above formula, we can get the acoustic equation. As mentioned above, all the previous studies need to transform CPML equation into a new second-order partial differential equation that in the original coordinate. Instead of rewriting the CPML equation (5), we propose to directly simulate equation (5) by FD method. To elaborate our FD scheme, we first introduce four derivative operators, the forward stagger-grid derivative operator \( S_x \), backward stagger-grid derivative operator \( S_x^- \), regular-grid first-order derivative operator \( R_x \) and regular-grid second-order derivative operator \( R_x^- \):

\[ S_x u_{i,j} = \frac{1}{\Delta x} \sum_{l=1}^{L} C_S \left( u_{i+l,j} - u_{i-l,j} \right), \quad S_x^- u_{i,j} = \frac{1}{\Delta x} \sum_{l=1}^{L} C_S \left( u_{i+l,j} - u_{i-l,j} \right) \]

\[ R_x u_{i,j} = \frac{1}{\Delta x} \left( C_R u_{i,j} + \frac{1}{2} C_R \left( u_{i+1,j} + u_{i-1,j} \right) \right), \quad R_x^- u_{i,j} = \frac{1}{\Delta x} \left( C_R u_{i,j} + \frac{1}{2} C_R \left( u_{i+1,j} + u_{i-1,j} \right) \right) \]  

(6)

(7)

where \( u_{i,j} = u(i \Delta x, j \Delta z) \), \( \Delta x \) is the grid size along the \( x \)-axis, \( C_S \), \( C_R \) and \( C_R^2 \) indicate stagger-grid finite difference coefficients, regular-grid first derivative central difference coefficients and regular-grid second derivative central difference coefficients, respectively.

For a 2D rectangular area with PML boundary condition (Figure 1), we divide the calculation area into four blocks (A, B, C and D), and use the corresponding FD scheme in each block:

\[ \frac{\partial^2 u}{\partial x^2} = R_x^2 u_{i,j}, \quad \frac{\partial^2 u}{\partial y^2} = R_y^2 u_{i,j}, \quad \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} \right) = R_x R_y u_{i,j} \]  

(8)

\[ \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} \right) = S_x S_x u_{i,j}, \quad \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} \right) = S_x^- S_x^- u_{i,j} \]  

(9)

\[ \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} \right) = R_x R_x^- u_{i,j}, \quad \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} \right) = R_x^- R_x^- u_{i,j} \]  

(10)

where \( S_x, S_x^- \) and \( R_x, R_x^- \) are the second-order and mixed second-order partial derivative, respectively, in the PML domain and can be calculated:

\[ S_x u_{i,j} = \frac{1}{\kappa_x} S_x \left( S_x u_{i,j} \right) + \psi_x \left( S_x u_{i,j} \right), \quad \psi_x \left( S_x u_{i,j} \right) = b_x \psi_x^{(n-1)} \left( S_x u_{i,j} \right) + c_x \left( S_x u_{i,j} \right) \]  

(11)

\[ S_x^- u_{i,j} = \frac{1}{\kappa_x} S_x^- \left( S_x^- u_{i,j} \right) + \psi_x \left( S_x^- u_{i,j} \right), \quad \psi_x \left( S_x^- u_{i,j} \right) = b_x \psi_x^{(n-1)} \left( S_x^- u_{i,j} \right) + c_x \left( S_x^- u_{i,j} \right) \]  

(12)

\[ R_x R_x^- u_{i,j} = \frac{1}{\kappa_x} R_x \left( R_x u_{i,j} \right) + \psi_x \left( R_x u_{i,j} \right), \quad \psi_x \left( R_x u_{i,j} \right) = b_x \psi_x^{(n-1)} \left( R_x u_{i,j} \right) + c_x \left( R_x u_{i,j} \right) \]  

(13)
\[ R^1 u_{i,j} = \frac{1}{k_z} R^1 u_{i,j} + \psi^*(u), \quad \psi^*(u) = b_y \psi^{n+1}(u) + c_z R^1 u_{i,j}. \] (14)

**Figure 1** Finite difference scheme of the second-order and mixed second-order spatial derivative in computational domain B and PML domain A, C, and D.

**Examples**

In this section, we show two numerical results of TS-CPML boundary condition. Figure 2 shows the absorbing performance for the second-order acoustic wave equation. Figure 3 shows the absorbing performance for the second-order elastic wave equation.

**Figure 2** (a) The 2-D heterogeneous velocity model with a dimension of 15km × 15km, consisting of 300 cells in each direction. The thin dashed lines separate the 270-cell computational domain and the 30-cell PML domain. (b) Snapshots of the acoustic wavefields at 1.5, 3.0, 4.5 and 10s computed with split-field PML and TS-CPML are shown in the top and bottom rows. The black triangle indicates the location of the hypothetic receiver that records seismogram. (c) Synthetic seismograms at receiver location computed with split-field PML and TS-CPML are shown in blue dashed-dotted and red dotted lines, respectively. For comparison, the reference seismogram is also shown in black solid line. Inset shows the enlarged seismograms. Note that reflections from the split-field PML are much larger than those from the TS-CPML.
Figure 3 Snapshots of the wavefields at 0.3, 0.5, 0.8 and 1.9s. (a) Horizontal and Vertical displacement snapshots of the second-order displacement elastic wave equation using the TS-CPML boundary condition. The black triangle indicates the location of the hypothetic receiver that records seismogram to quantify the relative errors. (b) The difference of horizontal displacement seismograms and vertical displacement seismograms between the reference solution and the TS-CPML synthetics at the receiver indicated by black reverse triangle in Figure 3(a) computed with a 10-cell thick, 20-cell thick, and 30-cell thick PML layers are shown in brown dotted, blue dashed and red solid lines, respectively. Note that the difference decreases with increasing PML layer thickness.

Conclusions

We propose a new implementation of unsplit CPML for the second-order wave equation system. The proposed method does not need to introduce auxiliary terms to rewrite the second-order CPML wave equation and we can simulate the second-order acoustic or elastic CPML equation directly in the time domain. The number of convolution terms of proposed method is less than the conventional CPML implementation to the second-order wave equation. The implementation for the second-order wave equation system is simple and efficient based on recursive convolution method.

References


