Introduction
The finite-difference (FD) method is an essential component in seismic modeling. It is vital in all full-wave algorithms as well as for inversion problems such as full waveform inversion (FWI) (Tarantola, 1984) or reverse-time migration (RTM) (Baysal et al., 1983). Furthermore, this is generally the most computationally intensive component of seismic imaging and inversion schemes and the cost is problematic in many situations. Thus, any effort to speed up the computation will have a big impact. One particular way to reduce the cost of finite difference simulations is by using local solvers (Robertsson and Chapman, 2000; van Manen et al., 2007; Willemsen and Malcolm, 2017). These algorithms reduce the size of the computational domain to small areas. Local solvers are divided into two categories. In the first type of solver, boundary conditions are used to immerse the local domain in an extended domain. In the other type of solver, wave propagation is first performed in the background model and then used as an input to obtain the wavefield inside the local area; in this case the wavefield is injected as a set of monopole and dipole sources (e.g., Altemar and Karal, 1968; Robertsson and Chapman, 2000; Broggini et al., 2017; Jaimes Osorio and Malcolm, 2018). In this study, we use the second category. The wavefield injection method is known as multiple point sources (MPS) and it has been extensively studied (Morse and Feshbach, 1953). For instance, Masson et al. (2014) and Amundsen and Robertsson (2014) show that wavefield injection using MPS is equivalent to the boundary condition approach described by Robertsson and Chapman (2000). Vasmel and Robertsson (2016) discuss implementation of the MPS method in a standard staggered FD algorithm for acoustic wave equation and its application to RTM. In addition, Koene and Robertsson (2018) present a method to obtain the free-surface response of all wavefield constituents using two additional, localized, finite-difference simulations in both the acoustic and elastic domains. In this paper, we show how to implement the MPS method in a centred staggered FD scheme for isotropic elastic wave equation. The MPS method can be used in algorithms that utilize elastic to elastic local solvers. Our method is designed as an add-on to existing codes, allowing any finite-difference solver to be used as a local wave-equation solver. We implement the MPS in such a way that the resulting code has the same order of accuracy as the original FD modeling code. We implement our algorithm using the Devito tool (Louboutin et al., 2017).

Methodology: Injection of wavefields
We first consider elastic wave propagation in an isotropic, heterogeneous medium. Using the elastic representation theorems, we can determine the displacement $u_n(x, t)$ within a volume $V$ from measurements of displacement $u_n(x', t)$ and normal traction $\tau_{ij}(x', t)n_j$ on an enclosing surface $S$, (e.g. De Hoop, 1958; Aki and Richards, 2002) via:

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\begin{align*}
    u_n(x, t) & = \oint_S G_{in}(x, x', t) * \tau_{ij}(x', t)n_j dS' - \oint_S u_i(x', t) * C_{ijkl} \partial_l G_{kn}(x, x', t)n_j dS', \\
    \end{align*}
$$

where $u_n(x, t)$ is the n-th component of the displacement vector, $G_{in}(x, x', t)$ is the component of the Green's tensor that denotes the displacement at location $x$ in the direction $i$ due to a unit point force at $x'$ in the direction $n$, $*$ denotes temporal convolution, and $\partial_l$ represents the derivative with respect to the $x_l$ coordinate. The first term in equation (1) represents a distribution of monopole sources on $S$ with strength $\tau_{ij}(x', t)n_j$ and the second term represents a distribution of dipole sources on $S$ with strength $u_i(x', t)C_{ijkl}n_j$.

Equation (1) is an example of an elastic representation theorem. For this study, we implement the MPS method within a velocity-stress formulation of the discretized elastic wave equation. We derive a similar representation for velocity-stress formulation using the methods described by Wapenaar (2007). Equation (2) shows the final result of the monopole point-sources that are injected into the velocity-stress system (Virieux, 1986, equation (2)) on the boundary $S$. 

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    u_n(x, t) & = \oint_S G_{in}(x, x', t) * \tau_{ij}(x', t)n_j dS' - \oint_S u_i(x', t) * C_{ijkl} \partial_l G_{kn}(x, x', t)n_j dS', \\
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  f_x(x_s, t) &= \sum_{s=1}^{N_s} \frac{1}{\rho} [\tau_{sx} n_x + \tau_{sx} n_z] \delta_s \\
  f_z(x_s, t) &= \sum_{s=1}^{N_s} \frac{1}{\rho} [\tau_{sz} n_x + \tau_{sz} n_z] \delta_s \\
  h_{xx}(x_s, t) &= \sum_{s=1}^{N_s} [(\lambda + 2\mu) v_{x} n_x + \lambda v_{z} n_z] \delta_s \\
  h_{zz}(x_s, t) &= \sum_{s=1}^{N_s} [(\lambda + 2\mu) v_{z} n_x + \lambda v_{z} n_z] \delta_s \\
  h_{xz}(x_s, t) &= \sum_{s=1}^{N_s} \mu [v_x n_z + v_z n_x] \delta_s 
\end{align*}
\]

where \( f_x(x_s, t) \) and \( f_z(x_s, t) \) represent the point-sources injected as velocities in the first two equations. Notice that \( f_x(x_s, t) \) and \( f_z(x_s, t) \) depend on stresses recorded on \( S \). The parameters \( h_{xx}(x_s, t) \), \( h_{zz}(x_s, t) \) and \( h_{xz}(x_s, t) \) represent the point-sources injected as stresses in the last three equations, and they are combinations of velocity components recorded on \( S \). \( \delta_s \) denotes the delta-function centered at \( x_s \in S \) scaled by the discretization interval to ensure correct amplitude, and \( N_s \) represents the number of sources equal to the number of grid points on \( S \).

To solve the elastic wave equation we use a centred staggered FD grid with a second order accuracy in time and space (e.g., Virieux 1986). Figure 1 shows the boundary \( S \) intersecting the staggered grid. In this figure, \( \tau_{xx} \) and \( \tau_{zz} \) can be recorded exactly on \( S \) (circles). On the other hand, due to the staggered grid configuration, \( v_x, v_z \) and \( \tau_{xz} \) are computed on the nodes offset from the boundary \( S \). To calculate their values on \( S \), we use bicubic interpolation from the nodes surrounding the desired location. The number of nodes used in the interpolation depends on the length of the FD stencil and is chosen to match the accuracy of the FD scheme. To clarify, all the sources (equation 2) are injected at the position of the stresses \( \tau_{xx} \) and \( \tau_{zz} \) (circles), however, monopole sources in the equations corresponding to velocities \( v_x \) and stress \( \tau_{xz} \) are distributed during the injection from the interpolation position on \( S \) to the corresponding grid positions (triangles and squares).

**Figure 1** Grid locations for a staggered FD scheme. Red square represents the location of the wavefield in the staggered elastic FD grid points.

**Numerical Examples**

We demonstrate the MPS implementation on a part of the SEAM East-West 2D elastic model. This is a deepwater subsalt Earth model based on the Gulf of Mexico (Fehler and Keliher 2011). The full \( V_p \) velocity model is shown in figure 2a. We define a small area bounded by the blue square as our full model. It is shown in more detail in Figure 2b. Figure 2b also shows the boundary \( S \) where the injection occurs (red points). To reconstruct the original wavefield inside the injection boundary \( S \), we save five different wavefields: \( v_x(x_s, t), v_z(x_s, t), \tau_{xx}(x_s, t), \tau_{zz}(x_s, t), \) and \( \tau_{xz}(x_s, t) \) on the boundary \( S \).

Figures 3a and 3b show the shot gathers generated by using the elastic solver in the full domain and by the injection of the secondary sources (equation 2). Figure 3c shows traces at \( x = 5000 \text{ m} \) corresponding to the two scenarios in figures 3a and 3b. We can see that the traces match perfectly. Figures 4a and 4b show the stresses \( \tau_{xx} \) at time \( t = 0.964 \text{s} \) modeled in the full domain and with the
wavefield injection respectively. Figure 4c shows the difference between the two previous stresses with a percentage of error estimated equal to $8.06 \times 10^{-05}\%$.

Figure 2 (a) SEAM elastic model, based on the gulf of Mexico. The blue square represents the domain we use to implement the injection/reconstruction of the wavefield. (b) The zoomed domain selected (blue points in 2a). Red points represent the injection positions of the secondary sources, the black points are the receiver positions between $x = 4000 – 6000$ m in distance and $z = 1000$ m in depth, and the yellow point is the source position at $x = 5000$ m and $z = 200$ m.

Figure 3 (a) Shot gather for the source at $x = 5000$ m computed in the full domain and (b) the shot gather reconstructed by the injection of secondary sources (equation 2, red points, Figure 2). (c) Trace corresponding to the two scenarios in (a) and (b) at $x = 5000$ m.

Figure 4 Snapshots of stress $\tau_{x,x}$ at time $t=0.964$ s ($nt=760$): (a) computed in the full domain, (b) computed by injection of the wavefield, and (c) difference between (a) and (b).

Conclusions
We have described the implementation of the MPS method in a centred staggered FD scheme for the elastic isotropic wave equation in the velocity-stress formulation, and its applicability in an elastic to elastic local solver. We show that using only one surface and five wavefields we are able to accurately retrieve the wavefield within a subdomain. The MPS method thus requires significantly less storage than alternative reconstruction methods such as the method of Robertsson and Chapman (2000). Because the injection of the wavefield takes place by specifying point sources rather than by changing the parameters of the finite difference stencil, the implementation with any existing seismic modeling tool is straightforward.

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References


