Reflection of inhomogeneous seismic waves at the free surface of an effective Biot solid

Introduction

Most materials near the Earth’s surface, especially reservoir rocks are anelastic with significant intrinsic energy absorption. The seismic waves propagating in such materials are generally inhomogeneous waves (i.e. different directions of attenuation and propagation). The boundary conditions at an interface separating dissimilar anelastic materials can only be satisfied by considering inhomogeneous (not homogeneous) waves. Effective Biot theory (Pride et al., 2004) has been extended from classic Biot theory as the most popular theory to describe wave propagation in a porous solid (e.g., a reservoir). Liu et al. (2018) upscaled and generalize effective Biot theory to a poroviscoelastic model (with highly dissipative S-waves) for which the anelastic moduli are represented by the relaxation functions of the generalized fractional Zener model. Based on this model, Liu and Greenhalgh (2019) investigated the energy ratios (how the incident energy is partitioned amongst the reflected inhomogeneous P and S waves) at the free surface of an effective Biot material. Since the inhomogeneity parameter $D$ ($0 \leq D \leq \infty$), which represents the degree of wave inhomogeneity degree, is incidence/reflection angle dependent, the dissipation factor $Q^{-1}$ and the phase velocity of the reflected waves, which are strongly dependent on $D$, must also be angle dependent. However, $D$ of the reflected waves are not normally known. Therefore, their phase velocity and $Q^{-1}$ are not known either. To the best of our knowledge, there is no publication which specifically has investigated $D$ of reflected/transmitted inhomogeneous waves. This gap in knowledge will be partially filled in this paper by considering the free surface reflections of incident P and S waves in an isotropic effective Biot material as a relatively simple case.

Extended Snell’s law

At the free surface of an effective Biot material, the solid particle displacement $u$ and the relative fluid displacement $w$ are the summations of the four wave components denoted with superscripts ($k = 0, 1, 2, 3$): the incident wave and the three reflected waves: $SV$, fast-$P$ and slow-$P$ waves (denoted as, $S$, $P_1$, $P_2$ respectively).

$$u^{(k)} = f^{(k)} \hat{u}^{(k)} \exp[i\omega(p^{(k)} \cdot r - t)], \quad w^{(k)} = g^{(k)} f^{(k)} \hat{u}^{(k)} \exp[i\omega(p^{(k)} \cdot r - t)].$$

(1)

Here, $\hat{u}^{(k)}$ or $(\hat{u}_x^{(k)}, 0, \hat{u}_z^{(k)})$ are the unit particle polarization vectors for each wave, $\hat{u}^{(k)} \cdot \hat{u}^{(k)} = 1$; $f^{(k)}$, complex amplitudes; $p^{(k)}$, complex slowness vector, written as $(p_x^{(k)}, 0, p_z^{(k)})$. For an inhomogeneous plane wave with an real inhomogeneity parameter $p$, $p$ is written as (Cerveny and Psencik, 2005):

$$p = c_n i D \hat{m} \quad \text{or} \quad p = p \hat{p} , \quad \hat{m} \cdot \hat{n} = 0 \quad \text{and} \quad \sigma^2 = p^2 + D^2 .$$

(2)

Substituting eq.(1) into the effective Biot wave equations, the quantity $p$ , the vector $\hat{u}$ and the coefficient $\xi$ and $f$ (using free surface conditions) are solved for $S$, $P_1$ and $P_2$ waves, respectively (Liu and Greenhalgh, 2019).

For a given incidence angle $\theta$ and an incident wave inhomogeneity parameter $D^{(0)}$, the complex Snell’s law shows $p_x^{(k)} = p_x$ , and we have

$$p_x^{(0)} = [\text{Re}[\sigma^{(0)}] + i \text{Im}[\sigma^{(0)}] \cos \theta - i D^{(0)} \sin \theta] p_x , \quad p_x^{(i)} = [\text{Re}[\sigma^{(0)}] + i \text{Im}[\sigma^{(0)}] \cos \theta + i D^{(0)} \sin \theta] p_x .$$

(3)

$$\hat{u}^{(i)} = (p_x^{(i)}, 0, -p_z^{(i)}) / p^{(i)} \quad \text{for} \quad S\text{-wave}; \quad \hat{u}^{(i)} = (p_x^{(i)}, 0, p_z^{(i)}) / p^{(i)} \quad \text{for} \quad P_1 \text{or} P_2 \text{wave} .$$

(4)

If the incidence angle $\theta = 0$, the real parts of $p_x$ and $p_x^{(i)}$ must be zero, which gives:

$$D^{(i)} = -D^{(0)} , \quad (\text{for} \quad \theta = 0, \text{and} \quad k = 1, 2, 3)$$

(5)

Eq. (5) specifies the inhomogeneity parameters of all reflected waves for the incident angle $\theta = 0$. This result provides a reference (starting) point to check or investigate the inhomogeneity parameters of the reflected waves for an arbitrary angle of the incident wave. For oblique (arbitrary) incidence, and the pure mode reflections (i.e. SS or PP), the eq. (3) and $p^{(i)} = p_x$ holds for all incident angles resulting in:

$$\theta^{(k)} = \theta , \quad D^{(k)} = -D^{(0)} , \quad \sigma^{(k)} = \sigma^{(0)} , \quad k \text{ is for the pure mode reflections.}$$

(6)
The same inhomogeneity parameters between the incident and reflected waves are indicated by this equation so the same dissipation factors \( Q^1 \) apply for incident and reflected waves. Since \( Q^1 \) of the incident waves are independent of incidence angle, so are the pure mode reflections. The quantities \( Q^1 \) of the mode-converted reflected waves are the challenge for which implicit and the explicit expressions of \( Q^1 \) are developed below.

**Dissipation factors and inhomogeneity parameters estimation**

According to Buchen (1971) and Carcione (2014), \( Q^1 \) can be defined with the time-averaged kinetic \( \langle T \rangle \), strain \( \langle V \rangle \), kinetic dissipated \( \langle D_v \rangle \) and strain dissipated \( \langle D_e \rangle \) energy densities as:

\[
Q^1 = \langle D_v \rangle + \langle D_e \rangle / \langle (T) + (V) \rangle, \tag{7}
\]

\[
\langle D_v \rangle = \frac{1}{4} \text{Im}(e^T \bullet P \bullet e^*), \quad \langle D_e \rangle = \frac{1}{2} \text{Im}(v^T \bullet R \bullet v), \quad \langle T \rangle = \frac{1}{4} \text{Re}(e^T \bullet P \bullet e^*), \quad \langle V \rangle = \frac{1}{4} \text{Re}(v^T \bullet R \bullet v). \tag{8}
\]

Here, \( P \) is the elasticity matrix and \( R \) the general density matrix Carcione (2014),

\[
P = \begin{bmatrix} H & \lambda & 0 & 0 & 0 & C \\ \lambda & H & 0 & 0 & 0 & C \\ 0 & 0 & H & 0 & 0 & C \\ 0 & 0 & 0 & G & 0 & 0 \\ 0 & 0 & 0 & 0 & G & 0 \\ C & C & 0 & 0 & 0 & M \end{bmatrix}, \quad R = \begin{bmatrix} \rho & 0 & 0 & \rho_r & 0 & 0 \\ 0 & \rho & 0 & 0 & \rho_r & 0 \\ 0 & 0 & \rho & 0 & 0 & \rho_r \\ \rho & 0 & 0 & \rho_r & 0 & 0 \\ 0 & \rho & 0 & 0 & \rho_r & 0 \\ 0 & 0 & \rho & 0 & 0 & \rho_r \end{bmatrix}.
\]

Here, \( \kappa', K' \) stand for grain, fluid bulk moduli; \( \phi \) porosity, \( \rho \) density, \( \rho_r \) and \( \eta \) are density and viscosity of the fluid, and \( K'(\omega) \) is the effective dynamic permeability; \( G(\omega) \) and \( K'(\omega) \) are the frequency dependent shear and bulk moduli of the solid frame (Pride, et al., 2004).

For SV- and P-waves in XZ plane, we have:

\[
e = (e_1, e_2, e_3, 0, 0, 0, -\zeta)^T, \quad e_1 = \bar{\partial}_x u_1, e_2 = \bar{\partial}_y u_2, e_3 = \bar{\partial}_z u_3, \quad \zeta = -div \mathbf{w}, \quad \mathbf{w} = \phi(u^T - u), \tag{9}
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With eqs (1) and (7 to 11), the implicit \( Q^1 \) expressions can be derived without knowing \( D \). The complex quadratic terms \( e^T \bullet P \bullet e^* \) and \( v^T \bullet R \bullet v \) in eq. (8) are derived for the wave type \( k \) \( (k = 0,1,2,3 \) for incident, and reflected S, P1, and P2 waves respectively) as:

\[
e^{T(k)} \bullet P \bullet e^{(k)} = A \left[ H + 2C \text{Re}(\zeta^{(k)}) \left[ \left[ \tilde{u}_{x}^{(k)} p_{x}^{(k)} \right]^{2} + \left[ \tilde{u}_{y}^{(k)} p_{y}^{(k)} \right]^{2} \right] + G \left[ \tilde{u}_{x}^{(k)} p_{x}^{(k)} + \tilde{u}_{y}^{(k)} p_{y}^{(k)} \right]^{2} \right] + M \left[ \tilde{u}_{x}^{(k)} p_{x}^{(k)} + \tilde{u}_{y}^{(k)} p_{y}^{(k)} \right]^{2} + 2 \lambda + 4C \text{Re}(\zeta^{(k)}) \text{Re}(\tilde{u}_{x}^{(k)} p_{x}^{(k)} + \tilde{u}_{y}^{(k)} p_{y}^{(k)}), \tag{10}
\]

\[
A = \alpha \left[ f^{(k)^2} - \beta(\sigma^{(k)^2} + \text{Im}(\sigma^{(k)})) \right].
\]

The explicit \( Q^1 \) expressions can be derived by assuming an incident wave with an arbitrary incidence angle (0) and amplitude \( f^{(0)} \). Using eqs. (2) and (4) to replace \( u^{(k)}, p_{x}^{(0)} \) and \( p_{y}^{(0)} \) with \( p \) and \( D \) in eqs (12) and (13) results in the explicit expressions \( Q_{s}^1 \) and \( Q_{p}^1 \) for S and P waves, respectively:

\[
Q_{s}^1 = \frac{2\rho^{2} \text{Im}(\tilde{p}^{2}) \left[ PD_{x}^{2} + D_{x}^{2} \right] - \left[ 4D' P D_{x}^{2} + \left[ PD_{x}^{2} + D_{x}^{2} \right] \text{Im}(G) \right]}{\left[ \rho + \rho_{r} \left[ 2 \text{Re}(\tilde{p}^{2}) + \left[ \left[ PD_{x}^{2} + D_{x}^{2} \right] \text{Im}(G) \right] + 2 \left[ 4D' P D_{x}^{2} + \left[ PD_{x}^{2} + D_{x}^{2} \right] \text{Im}(G) \right] \text{Re}(G) \right]}}, \quad PD_{x}^{2} = \left[ \rho^{(p)} \right]^{2} + D^{2}, \tag{11}
\]

\[
Q_{p}^1 = 2 \left( \frac{PD_{x}^{2} + D_{x}^{2}}{\rho + 2 \rho_{r} \left[ \text{Re}(\tilde{p}^{2}) + \left| \tilde{p}^{2} \right| \text{Re}(\tilde{p}) \right] + \left[ \left| \tilde{p}^{2} \right| \text{Re}(\tilde{p}) \right]} \text{Im}(H) + D^{2} \right), \tag{12}
\]

\[
\text{Re}(\tilde{p}) \left[ 2 \text{Re}(\tilde{p}) + D_{x}^{2} \right] \text{Im}(\tilde{p}) = \text{Im}(H) + D^{2} + \text{Im}(M), \tag{13}
\]

\[
-2D^{2} \text{Re}(\tilde{p}^{2}) + D^{2} \text{Im}(\tilde{p}) + 2 \left[ \text{Re}(\tilde{p}^{2}) + \left| \tilde{p}^{2} \right| \text{Re}(\tilde{p}) \right] \text{Im}(C) + \left| \tilde{p}^{2} \right| \text{Im}(M), \tag{14}
\]

\[
-2D^{2} \text{Re}(\tilde{p}^{2}) + D^{2} \text{Im}(\tilde{p}) + 2 \left[ \text{Re}(\tilde{p}^{2}) + \left| \tilde{p}^{2} \right| \text{Re}(\tilde{p}) \right] \text{Im}(C) + \left| \tilde{p}^{2} \right| \text{Im}(M), \tag{15}
\]
\[ VR_p = 4D^2 \left[ PD_p^2 \text{Re}(G) + \left(D^2 + PD_p^2 \right) \text{Re}(H) \right] - 2D^2 \left[ \text{Re}(PD_p^2 - D^2) + \left(D^2 \text{Re}(\lambda) + 2 \text{Re}\left(\rho^{(P)} \left| \rho^{(P)} \right|^2 \right) \text{Re}(C) + \left| \rho^{(P)} \left| \rho^{(P)} \right|^2 \right) \text{Re}(M) \right] . \]  

(18)

Note that here the P-wave can be the fast P-wave or the slow P-wave. For an infinite value of \( D \), then eqs (15) and (16) result in the same three limiting dissipation factors for S, P1 and P2-waves, i.e.,

\[ Q^{-1}(D = \text{inf}) = Q^{-1}(D = \text{inf}) = Q^{-1}(D = \text{inf}) = Q^{-1}(D = \text{inf}) = 2\text{Im}(G[\omega]/\text{Re}(G[\omega])). \]  

(19)

The limiting dissipation factors actually represent the frequency-dependent dissipation property of the shear modulus of the solid frame (the viscoelastic S-wave).

With eqs (3) and (4), \( D \) can be derived as a function of the complex slowness vector, referred to as the \( D-p \) equation, given by:

\[ D = \pm \frac{\text{Im}(p_x) \text{Re}(p_x) - \text{Im}(p_z) \text{Re}(p_z)}{/\sqrt{\text{Re}^2(p_x) + \text{Re}^2(p_z)}}. \]  

(20)

Dissipation factors \( Q^{-1} \) are estimated by two separate methods. Method 1 is to use eqs (7) to (13) without involving \( D \), and then extract \( Q^{-1} \) with the explicit eqs (15) and (16). Method 2 is to use eq. (20) to get \( D \) and then calculate \( Q^{-1} \) with eqs (15) and (16). The validity of the two methods can be verified by the equivalence of results. For the sake of convenience, we define a dimensionless quantity \( \hat{D} \) as the inhomogeneity coefficient equal to \( D \) multiplied with the unrelaxed phase velocity \( v_P^0 \) at \( \omega = \infty \) denoted as \( v_{Sp,Pl,Pl2a} \) for S, P1 and P2 waves, respectively: \( D = D_{vSp,Pl,Pl2a} \).

**Numerical examples of S-wave incidence at the free surface of an effective Biot solid**

The effective Biot material is chosen as the sample material in Liu et al. (2018, 2019 for detailed material parameters) in which \( G(\omega) \) and \( k^c(\omega) \) can be well approximated with the Cole-Cole model. For the example material, the P1-wave of the example is more attenuative than the S wave; the relaxed velocities (zero frequency) are \( v_{S0} = 1606 \text{ m/s}, v_{P0} = 3072 \text{ m/s} \) which roughly gives the critical angle of the S-P1 wave to be \( 30^\circ \). The unrelaxed phase velocities (infinite frequency) are \( v_{Su} = 1734 \text{ (m/s)}, v_{Plu} = 3746 \text{ (m/s)} \) and \( v_{P2u} = 776 \text{ (m/s)} \).

![Figure 1](image)

**Figure 1** (a) Energy ratios of the Biot reflected waves and the interaction energy for an incident S wave of frequency of 100 Hz with inhomogeneity coefficients 0.0001, 0.1 and 0.2; (b) Estimated dissipation factors of the reflected waves in Fig. 1(a); (c) Estimated inhomogeneity coefficients \( \hat{D}_{Sp} \) (upper diagram), phase velocity \( v_{Sp} \) (middle plot) of the S-P1-wave and the error function (lower diagram).

Fig. 1(a) shows the energy ratios (denoted as ER, see Liu et al., 2029 for the algorithm and equations) for an incident S wave at a frequency of 100 Hz. Results are given for three values of the incident wave inhomogeneity coefficient \( \hat{D} = 0.0001, 0.1 \) and 0.2. For the incidence angle exceeding the critical angle, the energy ratios of both S-S (upper) and S-P1 (middle) waves strongly depend on \( \hat{D} \) and have the ER values much larger than 1, which indicates the corresponding larger negative interaction energy ratios to ensure the energy conservation law (Liu et al., 2019). The S-P2 (lower) wave carries a very small amount of energy.
Fig. 1(b) shows the dissipation factors for the reflected S-S wave (1/Q_SS, upper diagram), S-P1 wave (1/Q_SP1, middle diagram) and the S-P2 wave (1/Q_SP2, lower diagram) corresponding to Fig. 1(a) and computed by eqs (7) to (14) without using $D$. As previously mentioned, the pure mode reflection, i.e. the SS wave here, has the same $Q^1$ value as the incident wave (eq. 6) and is independent of the incidence angle. The $Q^1$ values of the S-P1 (middle) clearly indicate the curves drop with increasing incidence angle, but decrease with increasing $\hat{D}$ of the incident wave. S-P2 waves (1/Q_SP2, lower) exhibit negligible change, indicating the same and constant $D$ as that of incident wave. Thus, we choose this curve to recover the inhomogeneity coefficient of the S-P1 wave.

Fig. 1(c) shows the estimated inhomogeneity coefficients ($\hat{\delta}_{sp1}$, upper diagram) and phase velocities ($v_{sp1}$, middle diagram) of the SP1 wave with the two aforementioned methods. For method 1 with a given value 1/Q_SP1, $\hat{\delta}_{sp1}$ is estimated by scanning $\hat{D}$ to yield the minimum error function $\text{error} = \text{abs} \left[ Q_{sp1}^{1}(\hat{D}) - Q_{sp1}^{1}(\theta) \right] \rightarrow 0$ given in the lower diagram. Method 2 can be used in a straightforward fashion. The phase velocity $v_{sp1}$ is obtained with eq. (2) and $\nu_{sp1}^{p_1} = 1/\text{Re}(\sigma)$. Fig. 1(c) shows that $\hat{\delta}_{sp1}$ increases with increasing incidence angle and $\hat{D}$ of the incident wave, which indicates the dissipation factor 1/Q_SP1 decreases with increasing $\hat{D}_{sp1}$ since it will finally reduce to its limiting dissipation factor, eq. (19). By contrast, $v_{sp1}^{p_1}$ decreases with increasing incidence angle and increasing $\hat{D}$.

Conclusions

By the complex Snell’s law, it is proven that under the vertical incidence, the inhomogeneity parameters of all reflected waves and the incident wave are equal. For oblique (arbitrary) incidence, the pure mode reflection waves have the same and incident angle-independent dissipation factors and inhomogeneity degree as the incident waves. Two sets of dissipation factor expressions (inexplicit and explicit) and the $D-p$ equation are derived for the three Biot waves, which are used to estimate the degree of inhomogeneity of the mode-converted reflected waves. The limiting dissipation factors are proved to be same for S, fast P- and slow P-waves, which helps to explain how dissipation factors change with the degree of wave inhomogeneity. The example of free surface reflections in an effective Biot solid is used to illustrate the new equations and algorithms.

Acknowledgements

We are grateful to King Fahd University of Petroleum & Minerals, KSA for supporting this research.

References


