Shearlet-based 3D inversion for frequency domain airborne electromagnetic data

Introduction

In the past few decades, airborne electromagnetic (AEM) inversion has developed rapidly and found many applications in mineral exploration, groundwater and environment investigations (Siemon et al., 2009; Kaminski et al., 2010; Chandra et al., 2019). Accordingly, the AEM modeling and inversion have also been extensively studied. In traditional regularization inversions, the regularization term is defined by the earth model parameters in the spatial domain, and generally only the model constraint between adjacent units is considered. Different inversion results can be obtained by changing the constraint term.

The sparse transformation has been applied to image processing, medical imaging, seismic data processing and many other fields in recent years. Among them, the wavelet transform has an excellent sparse representation for one-dimensional (1D) data, but it is not suitable for high-dimensional data due to the lack of directivity. The shearlet transform proposed by Kutyniok et al. (2014) not only has the characteristics of multi-scale analysis but also has the characteristics of multi-directionality, localization and compact support, which can optimally represent the edge of the target objects. It has good sparse representations for high-dimensional signals.

In this paper, we adopt the shearlet transform in our 3D EM inversions. We convert the resistivity model in the spatial domain to shearlet coefficients in the sparse domain and use them as constraints in the objective functional. The coefficients contain rich model information that can improve the model resolution. For the coarse scale, the shearlets capture the integral overview of the model, while for the fine scale, the shearlets capture the detailed features of the model, such as the edges of the complex structure. We verify the effectiveness of our method by inverting synthetic frequency-domain (FD) AEM data and will present the results for a survey dataset in our future presentation.

Method

Inversion strategy

In the shearlet-based algorithm, we calculate the sparse coefficients in the shearlet domain, and the inversion aims at finding a solution that accurately fit the observed data while keeping the coefficients as sparse as possible. The objective functional is defined as

$$\Phi = \| W_d (d^{obs} - d^{ref}) \|_2^2 + \lambda \| c - c^{ref} \|,$$

where $d^{obs}$ is the observed data, $d^{ref}$ is the predicted data calculated from the model $m$. $W_d$ is a diagonal matrix, $c$ is the vector of shearlet coefficients, $c^{ref}$ is the shearlet coefficients of the reference model. The shearlet coefficient vector $c$ can be expressed as

$$c = W_c m,$$

where $W_c$ is the shearlet transform operator, $m$ is the resistivity model in the spatial domain. 3-D shearlet transform is used to convert the resistivity model $m$ into a series of coefficients $c$. According to the chain rule, the sensitivity matrix $J_c$ in the shearlet domain can be expressed as

$$J_c = JW_c^{-1},$$

where $J$ is the sensitivity matrix in the spatial domain, $W_c^{-1}$ is the inverse shearlet transform operator.

In the following, we select GN method to iteratively minimize the objective functional. For the $n^{th}$ iteration, the linear equations system for the solution of the inverse problem is

$$(W_c J^T W_c J W_c^{-1} + \lambda \times R) \delta c$$

$$= W_c J^T W_c J W_c^{-1} (d^{obs} - d^{n-1}) -$$

$$+ \lambda \times R (c^{ref} - c^{n-1})$$

In above equation, $d^{n-1}$ is the data vector calculated from the previous model $m^{n-1}$, $\delta c = c^n - c^{n-1}$ is the coefficients updates of the $n^{th}$ iteration. As for diagonal matrix $R_c$ and $R_m$, the diagonal elements can be expressed as
\[
R_{ii} = \left( x_i^2 + e_i^2 \right)^{1/2},
\]
\[
\chi_i = c_i^{\text{ref}} - c_i^{n-1}.
\]
The model update in the spatial domain is expressed as
\[
m^i = m^{n-1} + sW^{-1}\delta c,
\]
where \(s\) is the scaling factor determined by a linear search (Haber, 2014).

Shearlet transform

Like the wavelet transform, the shearlet transform is also a multiscale analysis method that converts the model parameters into the shearlet coefficients at different scales. Whereas the basis function for the wavelet transform is “block basis” that is isotropic, the basis function for the shearlet transform adopts “wedge base” that guarantees the multiple directions.

To better understand the superiority of shearlet-based inversion, we briefly describe the construction of discrete shearlet system and the properties of shearlet transform. In the 2-D case, the non-separable shearlet generator is used to construct the digital shearlet (Lim, 2013), namely
\[
\hat{\psi}(\xi) = P(\xi_1/2, \xi_2)\hat{\psi}_{\text{sep}}(\xi),
\]
\[
\psi_{\text{sep}} = \psi_1 \otimes \phi_1,
\]
where \(\xi_1\) and \(\xi_2\) are parameters of FD, the trigonometric polynomial \(P\) is a 2-D directional filter, \(\psi_1\) is the wavelet function with enough vanishing and \(\phi_1\) is an associated scaling function, both \(\psi_1\) and \(\phi_1\) are sufficiently smooth. The 2-D shearlets appear as wedge-shaped in the FD as shown in Figure 1. For the 3-D case, the digital shearlet filter can be constructed by two 2-D digital shearlet filters (Kutyniok et al., 2014).

\[
\psi(\xi) = \left( P(\xi_1/2, \xi_2)\hat{\psi}_1(\xi_1)\hat{\phi}_1(\xi_2) \right) \cdot \left( P(\xi_3/2, \xi_3)\hat{\psi}_1(\xi_3) \right).
\]

**Figure 1** Shearlet generator \(\psi\) in frequency-domain. (modified from Lim, 2013) (a) Essential support of \(P(\xi_1/2, \xi_2)\hat{\psi}_1(\xi_1)\hat{\phi}_1(\xi_2)\); (b) essential support of \(P(\xi_3/2, \xi_3)\hat{\psi}_1(\xi_3)\); (c) essential support of \(\psi\).

The following equation shows how to calculate the shearlet coefficients, i.e.
\[
c(i) := \text{IFFT} \left( \hat{\psi}_{j,k}^{3D}(i) \cdot m_{\text{freq}} \right),
\]
where \(i\) represents the number of shearlet filters. Firstly, we use Fourier transform to compute frequency representation of the resistivity model \(m_{\text{freq}}\). Then, we calculate the pointwise multiplication for each conjugate of shearlet filter with \(m_{\text{freq}}\). Finally, the inverse Fourier transform is used to obtain the shearlet coefficients.

Example

We take the FD Helicopter Electromagnetic (HEM) system as example. This system has three horizontal coplanar coils and two vertical coaxial coils. The model we designed is a regular hexahedron with 64 cells in the \(x\)- and \(y\)-direction, respectively. In the \(x\)- and \(y\)-direction, the width of the grid is 15 m, extending 960 m long in total. In the \(z\)-direction, the total depth is about 1620 m. There is a total of 8.
survey lines, each of which has 19 measuring points. The survey line spacing is 100 m, while the survey point interval is 40 m. A total of 152 survey stations are evenly distributed within the central area.

For the synthetic model, we assume an inclined plate of 5 Ω·m embedded in a half-space of 100 Ω·m (Figure 2a). The initial model is assumed to be a half-space of 100 Ω·m. The initial regularization factor is 1000. For shearlet-based inversion, we set the shearlet system that contains 1 coarse-scale and 3 fine-scales. We add 5% Gaussian noise to the synthetic data.

Figure 2 and 3 show the inversion results from L2-norm and our shearlet-based inversion. The L2-norm inversion only delivers edge-smoothed results, the lower boundaries of the high- and low-resistivity anomalous body are not well recovered due to the smoothing regularization. In contrast, the shearlet-based inversion well recovers the anomalous body and delivers focused results, with the boundaries and convex and concave details being well delineated. It demonstrates that the shearlet-based inversion can better recover the boundaries of the abnormal body. This is largely due to the fact that the shearlet transform has an optimally sparse representation to the anomalous’ boundary.

**Figure 2** Section views in the x-direction of the synthetic example. (a) True model; (b) L2-norm inversion; (c) shearlet-based inversion.

**Figure 3** 3D views of the synthetic example. (a) True model; (b) L2-norm inversion; (c) shearlet-based inversion.

**Figure 4** (a) Data misfit rms; (b) objective functional; (c) regularization term.
Figure 4 shows the data misfit, the objective functional, and the regularization term for the two inversion algorithms. After 17 iterations, the data misfits for the L2-norm and shearlet-based inversions both drop to 1, and the regularization terms approach a constant value, meaning that they both converge well.

**Conclusion**

We have developed a well-resolved 3-D inversion method based on the shearlet transform to invert FD AEM data. Different from traditional model complexity measure, this new inversion algorithm uses the L1-norm minimization of the shearlet coefficients to get optimal solution. We have tested the synthetic data to compare the L2-norm and our shearlet-based inversion. From the results we found that our shearlet-based inversion can get higher resolution to boundaries of the targets than L2-norm method.

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**References**


