Introduction

The study of reflection coefficients benefits the in-situ property determination and also the acquisition system design for azimuthally anisotropic models. Conventionally, reflection coefficients are analyzed for the single interface. However, subsurface layers have finite thicknesses and wave scattering effects at the top and bottom of the layer interact with each other. For the normal incidence in the model composed of a thin layer embedded into a homogeneous space, Widess (1973) points out that the reflected wave amplitude greatly depends on the layer thickness. Rubino and Velis (2009) indicate the superiority of using reflection responses at various incidences to obtain the middle layer parameters nonlinearly. Compared with the exact reflection coefficients, approximations that depend on fewer model parameter contrasts can facilitate parameter estimations and sensitivity analyses. For the thin transversely isotropic layer with a vertical symmetry axis (VTI layer), Hao and Stovas (2017) derive the first-order reflectivity approximations based on an isotropic background medium. In this paper, more accurate approximations are derived with simpler analytical expressions.

Analyses of vertical fractures are of great interest in characterizing reservoirs. Orthorhombic (ORT) media typically characterize the anisotropy induced by a set of vertical parallel fractures or two sets of vertical and mutually orthogonal fractures embedded into a VTI host medium (Bakulin et al., 2000). We consider the ORT layer enclosed by two identical ORT half spaces to analyse the plane wave reflection coefficients normalized by vertical energy flux. We first calculate the exact reflection coefficients with the propagator matrix method (Fryer and Frazer, 1984) and then decompose the exact reflection coefficients into a series expansion. The series expansion relates the reflection coefficients for the single interface to the counterparts for the ORT layer. Therefore, under the weak-contrast assumption, the published reflection coefficient approximations for the single interface can be used directly to obtain the approximations for the ORT layer. We consider both anisotropic and isotropic background media. The proposed approximations are verified numerically.

Reflection Coefficient Modelling and Decomposition

We consider the model made up of an ORT layer embedded into an ORT space (Figure 1a). The layer thickness is assumed to be $H$. The ORT space and the ORT layer have aligned vertical symmetry planes that are parallel to the $[xz]$ and $[yz]$ planes in Cartesian coordinate $[xyz]$, respectively. The boundary condition at the upper ($z=0$) and lower interfaces ($z=H$) can be written as

$$
B|_{z=0} = B|_{z=-0} \quad \text{and} \quad B|_{z=H} = B|_{z=-H},
$$

where the vector $B$ is related to the wave amplitudes as follows (Fryer and Frazer, 1984),

$$
B = \frac{1}{\sqrt{2}} \begin{pmatrix} L_1 \\ L_2 \\ -L_2 \\ -L_1 \end{pmatrix} \times \begin{pmatrix} U \\ D \end{pmatrix},
$$

and $U = [U_p, U_S, U_{S1}, U_{S2}]$ and $D = [D_p, D_{S1}, D_{S2}]$. Symbols $U_k$ and $D_k$ are respectively the amplitudes for upward and downward wave mode $k$. The submatrices $L_1$ and $L_2$ are given in equations (A10) and (A11) in Jin and Stovas (2019b). Taking into account equation (1) and also the wave propagation in the ORT layer, we can obtain the reflection coefficients for the ORT layer as follows (Fryer and Frazer, 1984),

$$
R = \begin{pmatrix} R_{pp} & R_{pS} & R_{pS2} \\ R_{pS1} & R_{S1S1} & R_{S1S2} \\ R_{pS2} & R_{S2S1} & R_{S2S2} \end{pmatrix}=egin{pmatrix} P_{1i} e^{i\omega q_{x1}} & P_{1i}^{T} - P_{1i} e^{-i\omega q_{x1}} & P_{1i}^{T} + P_{1i} e^{i\omega q_{x1}} \\ P_{2i} e^{i\omega q_{x2}} & P_{2i}^{T} - P_{2i} e^{-i\omega q_{x2}} & P_{2i}^{T} + P_{2i} e^{i\omega q_{x2}} \end{pmatrix},
$$

where $\omega$ is the plane wave circular frequency and

$$
P_{1i} = [(L_{11}^{(1)})^T L_{12}^{(2)} + (L_{11}^{(2)})^T L_{12}^{(2)}]/2, \quad P_{2i} = [(L_{21}^{(1)})^T L_{21}^{(2)} - (L_{11}^{(1)})^T L_{22}^{(2)}]/2.
$$

The coefficients “$R_{ij}$” correspond to the reflected wave mode $i$ excited by the incident wave mode $j$. The superscripts “(1)” and “(2)” respectively denote the ORT half spaces and the ORT layer (Figure 1a). The vector $q_i^{(2)} = diag(q_i^{(2)}, q_{S1}, q_{S2})$ and the analytical expressions for vertical slownesses $q_i^{(2)}$ are given in Stovas (2017).
The exact reflection coefficients can be decomposed in terms of various orders of intrabed multiples in the ORT layer. The reflection and transmission (R/T) coefficients for the upper interface can be written as (Fryer and Frazer, 1984),

\[
\begin{align*}
\hat{R}_U &= -P_{12}^T P_{12}^{T*}, & \hat{T}_U &= P_{12}^{T*}, & \hat{R}_D &= P_{12} P_{12}^{-1} P_{12}^{-1}, & \hat{T}_D &= P_{12}^{-1},
\end{align*}
\]

where the subscripts “U” and “D” indicate the upward and downward incident waves, respectively. Taking into account equation (5), equation (3) can be rewritten as

\[
R = \hat{R}_D + \hat{T}_D e^{i\alpha q_{12}} \left[ I - (\hat{R}_U e^{i\alpha q_{12}})^2 \right] \hat{T}_U e^{i\alpha q_{12}} T_D,
\]

where coefficients \( \hat{R}_{U(D)} \) and \( \hat{T}_{U(D)} \) characterize the wave amplitude variations at the upper and lower interfaces, the factor \( e^{i\alpha q_{12}} \) corresponds to the wave propagation in the ORT layer. Equation (6) can further be written as a series expansion,

\[
R = \hat{R}_D + \hat{T}_D e^{i\alpha q_{12}} \left[ \sum_{j=0}^{\infty} (\hat{R}_U e^{i\alpha q_{12}})^2 (j+1) \hat{T}_D \right],
\]

where factors \( (\hat{R}_U e^{i\alpha q_{12}})^2 (j+1) \) characterize the propagation of \( j \)th-order intrabed multiples in the ORT layer (Figure 1b). The reflection coefficients account for the primary reflections from the upper and lower interfaces when equation (7) is truncated at \( j=0 \). Equation (7) gives essentially the same results as equation (3) but decomposes the reflection coefficients concerning various orders of intrabed multiples in the ORT layer. The reflection coefficients for the ORT layer (equation (7)) are directly related to the counterparts for the single interface (equation (5)). Subsequently, the published approximations for the single interface are used to obtain the counterparts for the ORT layer.

![Figure 1](image-url)

**Figure 1** (a) The sketch map for the ORT layer enclosed by two identical ORT half spaces; (b) the illustration for the intrabed multiples.

### Weak-contrast Approximations

We use the perturbation method to obtain the reflection coefficient approximations. The background medium parameters and the parameter discontinuities are defined by

\[
\bar{m} = (m^{(1)} + m^{(2)}) / 2 \quad \text{and} \quad \Delta m = m^{(2)} - m^{(1)},
\]

respectively, where “\( m^{(1)} \)” and “\( m^{(2)} \)” respectively indicate the parameters for the ORT half spaces and the middle ORT layer. We assume the ORT layer and the ORT half spaces have weak contrasts that \( |\Delta m / \bar{m}|<1 \). Because shear waves can have distinct polarization vectors in two ORT media that have weak contrasts, direct applying the perturbation method is inappropriate to obtain good reflection coefficient approximations for S1, S2 and converted waves. Jin and Stovas (2019a) use the pseudo wave projection (PWP) method to obtain the R/T approximations for the single interface bounded by two monoclinic half spaces. We extend the PWP method to approximate the reflection coefficients for the ORT layer. Two pseudo qSV and SH waves have polarization vectors given by

\[
u^{qSV} = \left( p_x, p_y, p_y (u_x S_1^1 u_x S_2^2 - u_y S_2^1 u_x S_1^1) + p_x (u_y S_2^1 u_x S_1^1 - u_y S_1^1 u_x S_2^2) \right) / u_x S_1^1 u_y S_2^2 - u_y S_1^1 u_x S_2^2 \quad \text{and} \quad \nu^{qSH} = (-p_y, p_x, 0),
\]

respectively, where \( p_x \) and \( p_y \) are horizontal slownesses in the \( x \) and \( y \) directions, respectively. Vectors \( \nu^{S1} = (u_x S_1^1, u_y S_1^1, u_z S_1^1) \) and \( \nu^{S2} = (u_x S_2^2, u_y S_2^2, u_z S_2^2) \) refer to the S1 and S2 wave polarization vectors, respectively. In terms of P, S1 and S2 waves, the reflection coefficients for the ORT layer (equation (3)) can be written as

\[
R = \Omega^{(1)} R_D^{(1)} (\Omega^{(1)})^{-1},
\]
where $R^M_{ij}$ refers to the reflection coefficients in terms of P, qSV and SH waves. The projection matrix is given by

$$
\Omega = \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos\Omega(u^{\text{qSV}}, u^{\text{S1}}) & \cos\Omega(u^{\text{SH}}, u^{\text{S1}}) \\
0 & \cos\Omega(u^{\text{qSV}}, u^{\text{S2}}) & \cos\Omega(u^{\text{SH}}, u^{\text{S2}})
\end{bmatrix}
$$

and

$$
\cos\Omega(u', u') = \frac{u'^T u'}{|u'|^2},
$$

where $I = \text{qSV}, \text{SH}$ and $J = \text{S1}, \text{S2}$. The superscript "(1)" indicates the calculation with the half space model parameters.

Because the reflection coefficients for P, S1 and S2 waves can be transformed from the counterparts for P, qSV and SH waves, we use the perturbation method to approximate the latter ones. Taking the R/T approximations for the single interface (Jin and Stovas, 2019a) into equation (7), we can obtain the first-order reflectivity approximations for the ORT layer, in terms of P, qSV and SH waves, given by

$$
R^M = (\bar{\Omega})^{-1} \times \begin{bmatrix}
R_{PP} & R_{PS1} & R_{PS2} \\
R_{S1P} & R_{S1S1} & R_{S1S2} \\
R_{S2P} & R_{S2S1} & R_{S2S2}
\end{bmatrix} \times \bar{\Omega},
$$

where

$$
A^n_g = 2 | g_n \sin[\omega H(\bar{q}_i + \bar{q}_j) / 2]|, \quad \Phi^n_g = [\omega H(\bar{q}_i + \bar{q}_j) + (\text{sgn}(g_n) - 2n)\pi] / 2, \quad n = 0,1,2,...
$$

The first-order scattering matrix elements $g_n$ for the upper interface, in terms of P, S1 and S2 waves, linearly depend on the model parameter contrasts and are given in Jin and Stovas (2019a). The vertical slowness $q_n$ is calculated with the background medium parameters for wave mode $k$. With the PWP method, we can obtain the first-order reflection coefficient approximations for P, S1 and S2 waves by taking equation (12) into equation (10). The first-order reflectivity approximations take into account the two primary reflected waves from the upper and lower interfaces, respectively (Figure 1b). The approximate PP wave amplitude for the ORT layer is a periodic function of the layer thickness and wave frequency. For $\omega Hq_{pp} \in (n\pi - \pi / 6, n\pi + \pi / 6)$, there is a destructive effect for amplitudes between the two primary reflected PP waves. For $\omega Hq_{pp} \in (n\pi + \pi / 6, n\pi + 5\pi / 6)$, there is a constructive effect for amplitudes between the two primary reflected PP waves. For the thin ORT layer with the thickness $H \ll \lambda$, where $\lambda$ refers to the wavelength of the propagating plane wave, the factor $n=0$ (equation (13)) and the two primary reflections only have destructive effects on each other. The phase shift between the incident and reflected waves approaches $\pm 90^\circ$ when the thin layer thickness decreases.

| Table 1 Parameters of the thin ORT layer and ORT half spaces. The stiffness coefficients are in GPa and the densities are in g/cm³. |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
|                 | $C_{11}$ | $C_{22}$ | $C_{33}$ | $C_{12}$ | $C_{13}$ | $C_{23}$ | $C_{44}$ | $C_{55}$ | $C_{66}$ | density |
| ORT layer       | 68.53     | 71.94     | 64.71     | 21.01     | 19.54     | 20.28     | 22.10     | 18.92     | 21.05     | 2.63     |
| half space      | 52.64     | 54.81     | 46.21     | 14.98     | 19.38     | 20.00     | 16.26     | 14.36     | 16.96     | 2.46     |

Model Tests

We implement numerical tests on an ORT model to validate the proposed approximations. The model parameters are listed in Table 1 for the ORT layer and ORT half spaces. The layer and half spaces are effective fractured VTI media (Bakulin et al., 2000). Because the reflection coefficients for P, S1 and S2 waves can be converted from the counterparts for P, qSV and SH waves using the PWP method (equation (10)), we consider the latter ones to test the proposed approximations. We consider the anisotropic background medium that has parameters averaged from equation (8) and also the isotropic background medium that is defined by Voigt-notation stiffnesses $\bar{C}_{j3}$, $\bar{C}_{55}$ and density averaged from equation (8). We assume the wave frequency to be 20Hz and the wave phase azimuth angle to be 30° taken from the x direction. To demonstrate the impacts of thin layer thicknesses on approximation accuracy, we consider different layer thicknesses that are no more than one-quarter of the P, S1 or S2
wave’s wavelength. Given that a reflection coefficient is a complex number that can be written as $Ae^{\Phi i}$, where $A$ is the amplitude and $\Phi$ denotes the phase factor, we show the exact amplitudes and phase factors and their approximation error magnitudes in Figures 2 and 3 for $R_{PP}$ and $R_{qSVP}$, respectively. When the thin layer thickness decreases, coefficients $R_{PP}$ and $R_{qSVP}$ have smaller amplitudes and their phase factors approach ±90°. The approximations have larger error magnitudes when the thin layer thickness increases, in terms of both amplitudes and phase factors. Overall, approximations with the anisotropic background medium (solid curves) have better accuracy than that based on the isotropic background medium (dashed curves).

![Figure 2](image1.png)  
**Figure 2** Exact amplitudes (a) and phase factors (b) for $R_{PP}$; approximation error for amplitudes (c) and phase factors (d) for $R_{PP}$. Curve colours denote different ORT layer thicknesses in the legend.

![Figure 3](image2.png)  
**Figure 3** Same with Figure 2 but for $R_{qSVP}$.

Conclusions
We consider the ORT model to characterize the vertically fractured medium and analyse the plane wave reflection coefficients for the ORT layer embedded into the ORT space. We decompose the exact reflection coefficients into a series expansion wherein the successive series terms correspond to different orders of intrabed multiples. Under the weak-contrast assumption, we adopt the PWP method and derive the first-order reflection coefficient approximations for the ORT layer. The approximations can give insights into the reflection responses. When the thin layer thickness decreases, the reflections have smaller amplitudes and the reflection phase factors approach ±90°. Numerical tests indicate better approximation accuracy with the anisotropic background medium than that using the isotropic background medium.

Acknowledgements
We would like to acknowledge the GAMES project and CSC for financial support.

References


