Wavefield simulation and absorbing boundary condition of staggered time integration method

Introduction

There has been a lot of work in solving the dispersion error of the numerical solution of the wave equation. Virieux (1984) introduced the staggered grid finite difference method (FDM), and proposed increasing the spatial order to improve the spatial accuracy, but the amount of calculation will increase accordingly. Dablain (1986) used the Lax-Wendroff method to improve the time accuracy of the wave equation to the fourth order of time. Klin (2010) pointed out that the Fourier pseudospectral method based on staggered grid has very good computational efficiency. Lee (2018) proposed a staggered time integration method to solve the wave equation in a recursive relationship, and the obtained value of the solution is almost non-dispersion in the time dimension, but the sponge absorbing layer is used in Lee’s original text to deal with boundary reflection, which cannot be widely used because of the large amount of calculation.

In this paper, we propose to combine Liao’s transmission formula (Liao et al. 1984; Liu and Sen 2010) and the staggered time integration simulation method for the second order wave equation (Lee, 2018) to absorb edge reflections. The simulation example shows that the weighted absorption boundary can absorb almost all boundary reflections with a small absorption thickness. Therefore, combined with our weighted mixed absorbing edge boundary, the staggered time integration forward method may have wide applications with seismic wave field simulation, such as migration and full-wavelfom inversion.

Method

The first-order acoustic wave equation is expressed as:

$$\frac{\partial}{\partial t} \begin{bmatrix} p \\ v \end{bmatrix} = \begin{bmatrix} 0 & -\rho c^2 \nabla^2 \\ -\rho c^2 \nabla^2 & 0 \end{bmatrix} \begin{bmatrix} p \\ v \end{bmatrix}$$

(1)

Expanding the pressure $p$ and particle velocity vector $v$ in the first-order acoustic wave equation with Taylor series to obtain the relationship between $v(t + \Delta t/2)$ and $v(t - \Delta t/2)$, $p(t + \Delta t)$ and $p(t - \Delta t)$, then replacing the time high-order derivative term with the spatial derivative, and giving initial condition $p(0) = p_0$ and $v(0) = 0$, we can get the recurrence relation of pressure with respect to time as

$$p(0) = p_0, \quad p(m\Delta t) = C(l)p_0,$$

$$p(m\Delta t + \Delta t) = 2C(l)p(m\Delta t) - p(m\Delta t - \Delta t), \quad m = 1, 2, \ldots$$

(2)

where

$$C(l) = 1 + 2\rho c^2 \nabla \cdot \frac{1}{\rho} \nabla$$

(3)

We set $c$ and $\rho$ as constants, and abbreviate $\rho c^2 \nabla \cdot \frac{1}{\rho} \nabla$ as $c^2 \nabla^2$, and then use the Fourier pseudo-spectrum method to calculate it

$$\rho c^2 \nabla \cdot \frac{1}{\rho} \nabla f = c^2 \nabla^2 f = c^2 \left( k_x^2 \mathcal{F}_x^{-1} \left( k_x^2 \mathcal{F}_x(f) \right) + k_z^2 \mathcal{F}_z^{-1} \left( k_z^2 \mathcal{F}_z(f) \right) \right).$$

(4)

The wavenumber domain $\tilde{C}(l, \theta)$ of $C(l)$ is expressed as

$$\tilde{C}(l, \theta) = 1 - 2 \left( \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!!} \theta^{2n+1} \right)^2, \quad \theta = c_k \Delta t, \quad k = \sqrt{k_x^2 + k_z^2}.$$ 

(5)

Therefore, the relationship between $p$ and time step number $m$ in the wavenumber domain can be obtained

$$\tilde{p}(m\Delta t) = T_m(C(l, \theta)) \tilde{p}_0, \quad m = 0, 1, 2, \ldots,$$

(6)
where T represents a polynomial.

Lee (2018) converted the high-order derivative of time to the calculation of the spatial derivative using the Fourier pseudospectral method, so that the numerical solution of the equation has almost no time dispersion error. But an efficient and effective absorbing boundary is necessary for an applicable simulation method. The sponge layer absorbing boundary Lee (2018) used requires a large amount of calculation, and the key to constructing the PML boundary condition is to split the wavefield into vertical and horizontal components. However, the derivation in formula (2) does not use the directional derivative, and formula (2) only has a recurrence relationship of three terms. As a result, PML is not suitable for equation (2). Therefore, we adopt a weighted mixed absorption boundary.

The 2D second-order acoustic wave equation can be written as

\[
\frac{1}{v(x)^2} \frac{\partial^2 p(x,t)}{\partial t^2} = \frac{\partial^2 p(x,t)}{\partial x^2} + \frac{\partial^2 p(x,t)}{\partial z^2},
\]

where \( p(x,t) \) and \( v(x) \) represents the pressure and velocity of seismic wave, respectively.

Liao’s transmitting formula of plane wave can be described as (taking the x-direction for an example, as shown in Figure 1)

\[
p(t + \Delta t, x_j) \approx \sum_{l=1}^{N} (-1)^{r+l} C_{l}^{N} T^r p',
\]

where the binomial coefficient \( C_{l}^{N} = \frac{N!}{(N-r)!r!} \), \( N \) represents the time order of the equation, \( T^r \) is a row matrix of \( (2r+1) \) elements

\[
T^r = [T_{r,1}, T_{r,2}, \ldots; T_{r,2r+1}].
\]

**Figure 1** Locations of computational points involved in equation (8).

It can be calculated by the following recursion relations:

\[
T' = T^r \begin{bmatrix}
T_{r-1,1} & T_{r-1,2} & \ldots & \ldots & T_{r-1,2r+1} & 0 & 0 \\
0 & T_{r-1,1} & T_{r-1,2} & \ldots & \ldots & T_{r-1,2r+1} & 0 \\
0 & 0 & T_{r-1,1} & T_{r-1,2} & \ldots & \ldots & T_{r-1,2r+1}
\end{bmatrix},
\]

where

\[
T^r = [T_{1,1}, T_{1,2}, T_{1,3}], T_{1,1} = (2-s)(1-s)/2, T_{1,2} = s(2-s), T_{1,3} = s(s-1)/2, s = \epsilon \Delta t/\Delta x,
\]

\[
P' = [p_{1,1}, p_{2,1}, \ldots; p_{2r+1,1}]^T, p_{i,j} = u(t_j, x_j), x_j = x_i - (i-1)\Delta x, t_j = t - (j-1)\Delta t.
\]

When we set \( N=2 \), then

\[
p(t + \Delta t, x_j) \approx C_{2}^{0} T^2 p' + (-1)^{2} C_{2}^{2} T^2 p'^2.
\]

After calculating and merging similar items, we have

\[
p(t + \Delta t, x_j) = 2T_{1,1}p(t, x_j) + 2T_{1,2}p(t, x_j - \Delta x) + 2T_{1,3}p(t, x_j - 2\Delta x) - T_{1,1}^2 p(t - \Delta t, x_j + \Delta x) - 2T_{1,1}T_{1,2}p(t - \Delta t, x_j - \Delta x)
\]

\[
-2(T_{1,1}T_{1,3} + T_{1,2}^2)p(t - \Delta t, x_j - 2\Delta x)
\]

\[
-2T_{1,3}p(t - \Delta t, x_j - 3\Delta x) - T_{1,3}^2 p(t - \Delta t, x_j - 4\Delta x).
\]

To sum up, when dealing with boundary reflection, we first calculate the acoustic wave equation to obtain the acoustic wave field \( p(m\Delta t) \) in the model area, then calculate another wave field \( p(t + \Delta t, x) \)
in the absorbing boundary area, and finally weight the input in the absorbing boundary area to obtain
the final wave field \( p_f = w_i p (m \Delta t) + (1 - w_i) p(t + \Delta t, x) \) in the absorbing boundary area, where \( w_i \) is
the linear weighting factor.

Examples

First, we simulate the wave field in a homogeneous medium model, the size of the mesh is \( 200 \times 200 \),
the spatial sampling interval is 10 m, and the time sampling interval is 1 ms. The source is a Richer
wavelet with a dominant frequency of 25 Hz, which is located in the center of the model. The seismic
wave velocity is 3000 m/s and the density is 1000 kg/m\(^3\). We set the time accuracy of the interleaved
time integral forward method to second order. Figure 2a and Figure 2b show the snapshots obtained
without absorbing in the \( x \)-direction and with the weighted mixed absorption boundary in both the \( x \)-
and \( z \)-directions. Figure 2c shows the waveform at (120, 120) node obtained using the absorbing
boundary. It can be seen from Figures 2a and 2b that when the incident wave reaches the boundary and
propagates outward, the weighted mixed absorption boundary can achieve a nearly perfect absorption
effect. The thickness of the absorption boundary is only 10 grid points, which is small.

\[ \text{(a)} \quad \text{(b)} \quad \text{(c)} \]

**Figure 2** (a) shows the simulation of the wave field without a weighted mixed absorption boundary in
the \( x \)-direction, (b) shows the simulation of the wave field with a weighted mixed absorption boundary
in both \( x \)- and \( z \)-directions, (c) shows the waveforms at (120, 120). The reflections from the boundaries
in (a) and (b) is well absorbed.

Then, we test the absorption effect of our weighted mixed absorption boundary in the forward
simulation of the BP gas model. Figure 3a shows the velocity of the BP gas model, whose size is 3.98
km \( \times \) 1.61 km. A source is located at (1.99 km, 0.15 km), the time interval is 1 ms. The rest parameters
are the same as those of the last example. Figure 3b shows a shot gather where the staggered time
integration method is used and a weighted mixed absorption boundary is added. It can be seen that the
absorption boundary can still achieve almost perfect absorption. The thickness of the absorption boundary is also 10 grid points.

**Figure 3** (a) shows the velocity of the BP gas model and (b) shows a shot gather.

### Conclusions

In this paper, the staggered time integral forward modeling method is combined with an efficient and effective absorbing boundary condition, so that it may be widely used in seismic wave field simulation. When dealing with boundary reflections, we weighted Liao’s plane wave transmission formula and acoustic wave field in the absorbing boundary area to alleviate the sharp change of the wave field in the boundary area. The results show that the weighted hybrid absorbing boundary can efficiently and effectively deal with the boundary reflections.

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### References


