Introduction

GLM (Gel’fand-Levitan-Marchenko) theory was introduced originally for scattering potential recovery of one-dimensional time-independent Schrödinger equation. Ware and Aki (1969) applied the theory to solve the 1-D elastic equation (for horizontal shear wave), but the solution is limited to the case of smooth impedance variation. Berryman and Greene (1980) introduced an integral for the K-operator to recover the low-wavenumber component of impedance. Coen (1981) formulated the GLM equation for oblique incidence to recover both velocity and density. The reduced GLM equation becomes an equation for refractive index, and the solution is also limited to smooth media. Howard (1983) developed a vector Marchenko equation based on the first order differential equation of motion and gave a solution of extracting density and velocity from reflection amplitudes of two incident angles. Rose (2002) gave a novel explanation to the integral kernel of the 1D Marchenko equation by introducing the concept of a time-domain focusing wavefield. Wapenaar et al. (2013) further considered spatial focusing wavefields and extended Marchenko’s method to 3D problems. Slob et al. (2014) discussed the application of Marchenko imaging to parameter (velocity and density) inversion for one-dimensional media. Wu (2018) applied the GLM theory and method to 1-D acoustic medium inversion including sharp boundaries by using the symbolic operation based on distribution theory and connect the method to the direct envelope inversion (DEI). Wu and He (2019) introduced the GLM equation for acoustic wave equation of oblique incident in layered media and gave some examples of simultaneous ρ-v inversion. In this paper, we further develop the method of velocity-density inversion based on GLM impedance solution and demonstrate the validity of the approach by stability analysis and numerical tests.

2. Impedance inversion based on GLM solution for Schrödinger impedance equation in layered acoustic media

In a general three-dimensional acoustic media, the wave equation can be written as

\[ \rho(x) \nabla \cdot \left( \frac{1}{\rho(x)} \nabla p(x,t) \right) - \frac{1}{c^2(x) \partial t^2} p(x,t) = 0. \]  

(1)

In layered acoustic media, the density \( \rho(z) \) and velocity \( c(z) \) are only \( z \)-dependent, but the pressure wavefield \( p(x,t) \) is a function of time and three-dimensional space. Therefore, acoustic equation for 1-D media with 3-D wavefield becomes

\[ \nabla^2 p(x,t) - \frac{1}{\rho(z)} \frac{\partial}{\partial z} \rho(z) \frac{\partial}{\partial z} p(x,t) - \frac{1}{c^2(z) \partial t^2} p(x,t) = 0. \]  

(2)

In the case of plane wave incidence, above equation can be reduced to a 1-D acoustic equation,

\[ \left( \frac{\partial^2}{\partial z^2} - \frac{1}{c^2(z) \partial t^2} \right) p(z, \omega) = 0, \]  

(3)

where \( c_\theta \) is an effective velocity depending on incident angle

\[ c_\theta(z) = \frac{c(z)}{\sqrt{1 - [c(z)/c_\theta]^2 \sin^2 \theta_0}}. \]  

(4)

It has been shown that 1-D acoustic wave equation such as Eq. (3) can be transformed into a Schrödinger impedance equation (Wu 2018; Wu and He, 2019),

\[ \left( \frac{\partial^2}{\partial \xi_\theta^2} - \frac{\partial^2}{\partial t^2} \right) \psi(\xi_\theta, t) = q_\theta(\xi_\theta) \psi(\xi_\theta, t) + s(\xi_\theta, t), \]  

(5)

where \( q_\theta \) is the scattering potential and \( s(\xi_\theta, t) \) is the real source. To derive the above Schrödinger equation, variable \( z \) has been changed to a travel-time related variable \( \xi_\theta \) by the Liouville transform and pressure field \( p \) was changed to an energy-flux normalized field \( \psi \) through
\( \psi(\xi_\theta,t) = p(\xi_\theta,t)\eta_\theta(\xi_\theta), \) where \( \eta_\theta(\xi_\theta) = 1/\sqrt{\rho(\xi_\theta)c_\theta(\xi_\theta)} \) is the impedance function. The impedance equation in Eq. (5) satisfies the zero-frequency Schrödinger equation, and therefore can be solved by the zero-frequency Jost solution of Schrödinger equation, leading to a solution

\[
\eta_\theta(\xi_\theta) = \frac{\eta_\theta(0)}{1 + \int_{-\xi_\theta}^{\xi_\theta} K(\xi_\theta,t)dt}.
\]  

(6)

where the kernel \( K(\xi_\theta,t) \) can be obtained by solving the GLM integral equation,

\[
K(\xi_\theta,t) = -R_\theta(t + \xi_\theta) - \int_{-\xi_\theta}^{\xi_\theta} K(\xi_\theta,\tau)R_\theta(t + \tau)d\tau,
\]

\[
R_\theta(t) \geq 0, -\xi_\theta \leq t \leq \xi_\theta
\]

(7)

where \( R_\theta(t) \) is the reflection record on the surface for oblique incidence.

3. Density and velocity inversion after obtaining impedance solutions for oblique incidence

If we measure reflection responses for both normal incidence \( R(t) \) and oblique incidence \( R_\theta(t) \), then we can obtain the corresponding impedance function \( \eta(\xi) \) and \( \eta_\theta(\xi_\theta) \). Then we can use both the amplitude ratio and the traveltime ratio of angle-dependent reflections for velocity inversion.

**Impedance ratio method of velocity estimation.** From Snell’s law we know that angle-dependence of reflection and refraction depends only on velocity structure so the \( \rho \) dependence is removed by taking the impedance ratio of two incident angles. Assuming the near-surface velocity \( c_0 \) is known, the velocity and thickness of any layer can be obtained independently as

\[
c_n = \frac{c_0}{\sin \theta_n} \sqrt{1 - \frac{\eta_{\theta}(z_n)}{\eta^2(z_n)}}, \quad \Delta z_n = c_n \Delta \xi_n.
\]

(8)

The density is easy to obtain with known velocity by \( \rho(z) = 1/c(z)\eta^2(z) \).

**Traveltime method for velocity estimation based on focusing principle.** Traveltimes between layer \( n \) and layer \( n+1 \) can be measured from impedance curves for both normal and oblique incidences,

\[
\Delta \xi(n) = \xi(n+1) - \xi(n), \quad \Delta \xi_\theta(n) = \xi_\theta(n+1) - \xi_\theta(n)
\]

(9)

The corresponding spatial distances are

\[
\Delta z(n) = c_n \Delta \xi(n), \quad \Delta z_\theta(n) = c_n \Delta \xi_\theta(n) / \cos \theta_n
\]

(10)

By the focusing principle, the pulses from normal incidence and oblique incidence should focus to the same spatial location, so \( \Delta z(n) = \Delta z_\theta(n) \). Velocity of the \( n \)th layer can be obtained as

\[
\overline{c}_n = \frac{c_0}{\sin \theta_n} \sqrt{1 - \left(\frac{\Delta z_\theta(n)}{\Delta \xi(n)}\right)^2}.
\]

(11)

This \( \overline{c}_n \) estimate may have different noise-resistance property than that from the amplitude ratio.

4. Stability analysis

First, we discuss the impedance inversion as a function of traveltime. Historically, layer-stripping method has been used to invert for local reflection coefficient (LRC) (e.g. Yagle 1996), and then recover the impedance profile from LRC. Lately, Marchenko imaging method is introduced for LRC mapping. Slob et al. (2014) discussed its application to parameter inversion for one-dimensional media. In the latter approach, LRC is determined by taking the amplitude of upgoing focusing function at the focusing time and the up- and down-going focusing functions are solved by the coupled Marchenko equations. In this way it avoids the procedure of layer-stripping and therefore becomes more stable. However, LRC is calculated using the instant value of upgoing focusing function and traveltime error of first arrival determination may bring errors to LRC. The third method, the integral method, as we used in this paper, is to use the direct solution of inverse Schrödinger impedance equation. As shown in (6) and (7), firstly the focusing function (or the kernel function) \( K(\xi,t) \) is solved and then the impedance function \( \eta(\xi) \) is calculated by taking integral. In this approach, the intermediate step of finding LRC \( r(\xi) \) is avoided. The integral method of finding impedance directly at each step will not suffer from the error propagation in the indirect methods of
deriving impedance from $r(\xi)$. Note that the integral method works on the impulsive data $R(t)$ and does not use the waveform data directly. Waveform data, such as generated by a source with Ricker wavelet, do not have D.C. component. Envelope data (with polarity) have similar property as impulsive data, therefore can be used in the integral method (see the related discussion in Wu, 2018).

Secondly, we discuss the stability problem of parameter (mainly the velocity) inversion from the resulted impedance profiles. From (8) and (11) we see that both the amplitude ratio method and the travelt ime ratio method are not layer-stripping in nature. The velocity estimate does not depend on the velocity value of the previous step and depends only on the amplitudes or traveltimes of the current arrivals of different incident angles. This avoids the error propagation of velocity estimation. However, the reflector positions suffer from the error-propagation. We need perform some kind of coherent stacking or amplitude-travelt ime joint inversion to increase the stability and noise resistance.

5. Numerical tests

We consider here a layered half space with an impedance layer having both velocity and density perturbations on top of a pure density perturbation layer. The frequency band for generating the synthetic seismograms is from 0 to 41 Hz. The time-domain impulse reflection series for normal incidence $R(t)$ and for oblique incidence $R_\alpha(t)$ are generated by the fast Fourier transform (FFT) as shown in the left part of Fig. 1. Reconstructed impedance functions by solving the GLM impedance equation are shown in the middle of Fig. 1: $\eta(\xi)$ (black line, $\theta = 0$) and $\eta(\xi)$ (red line, $\theta = 20^\circ$). On the right are reconstructed $\rho$ (top) and $c$ (bottom) profiles using amplitude ratio method. The results by the travelt ime ratio method has similar accuracy. We see that for noise-free data the density and velocity profiles are recovered with satisfactory accuracy.

Now we test the noisy data. We add white Gaussian noise to the reflection records (examples are shown in Fig 2 for 3% N/S amplitude ratio and Fig 3 for 5%). First, plane wave synthesis is similar to a coherent stack, so can reduce the noise accordingly. Here we assume the receiver array has 500 receivers. Then we show the GLM impedance inversion for the noisy data (in the middle panels of Fig 2 and 3). Finally, the recovered $\rho$ (top) and $c$ (bottom) profiles are shown on the right panels of Fig 2 and 3. We see that, as we discussed in the stability analysis, although the velocity inversion is a stable process. However, starting from 5% noise level, the recovered density profile becomes less accurate in the deep part, and has more artifacts. This is due to the fact that in the deep part beneath the last reflector, there is no primary reflections in the data and the multiples have much weaker signals than the primaries. Therefore, the S/N is deteriorated for the deep part of the inverted kernel $K(\xi,t)$ and the related impedance curve $\eta(\xi)$. Further study on the stable inversion for noisy data, such as the coherent stack and reflection-refraction (amplitude-travelt ime) joint inversion for each layer, is the next step of investigation.

Conclusion

We formulate the GLM (Gel’fand-Levitan-Marchenko) impedance solutions for layered media with different incident angles and developed two methods for velocity-density inversion: amplitude ratio and travelt ime ratio methods for angle-dependent impedance curves. Stability analysis and numerical
tests demonstrated that both the impedance inversion and the subsequent velocity-density inversion are stable inversions. However, noise resistance property of the method needs to be further studied.

Fig 2 Noisy data test for the case of 3% noise level. Captions are same as in Figure 1

Fig 3 Noisy data test for the case of 5% noise level. Captions are same as in Figure 1

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References


