Introduction

Accurately simulating and understanding the propagation of seismic wave is of great significance for seismic inversion. Due to its high-precision and low frequency characteristics, discontinuous Galerkin finite element method (DG-FEM) has been extensively studied, and it is also easy to realize parallel computation.

In the DG-FEM method, there are two choices, modal and nodal approach respectively. The modal approach would present solution by its values at the anchor points of Lagrange polynomials used to approximate the solution within each element. Modal DG-FEM methods for elastic wave propagation was first published by Käser and Dumbser (2006) for the 2D case, later extended to 3D case, viscoelasticity case, anisotropy case and poroelasticity case. Nodal DG-FEM methods for seismic wave modeling were first developed by Delcourte et al. (2009) and Etienne et al. (2010) for the isotropic, elastic case. However, the open boundary conditions used by Käser and Dumbser (2006) have strong artificial reflections. Etienne (2010) and L. Lambrecht (2017) combined CPML and NPML with Nodal DG-FEM, respectively, and they both achieved good absorption effects. Unsplit PML based on modal DG-FEM has not been studied much.

In this study, modal DG-FEM is used for 2D elastic medium seismic wave simulation. ADE CFS-PML (auxiliary differential equations complex frequency shifted PML) proposed by Zhang and Shen (2010) is modified and combined with DG-FEM to achieve good results. The modified method can be solved in the PML layer using the DG-FEM unified format, and the auxiliary differential equation becomes a first-order ordinary differential equation, which is convenient to solve. The results are compared with the analytical solutions obtained by generalized reflection and transmission (GRT) coefficient method (Chen, 1993), proving the reliability of our method.

Method

The 2-D elastic wave equation for an isotropic medium in velocity-stress formulation can be written as compact form

\[
\frac{\partial \mathbf{u}}{\partial t} = A(\vec{x}) \frac{\partial \mathbf{u}}{\partial x} + B(\vec{x}) \frac{\partial \mathbf{u}}{\partial z} + \mathbf{S}(\vec{x}),
\]

where \( \mathbf{u} = [\sigma_{xx}, \sigma_{zz}, \sigma_{xy}, \sigma_{xz}, \sigma_{yz}], \) \( \sigma_{xx}, \sigma_{zz} \) are the normal stress, \( \sigma_{xy}, \sigma_{xz}, \sigma_{yz} \) are the shear stress, \( v_x, v_z \) are the horizontal and vertical components respectively, \( \mathbf{S}(\vec{x}) \) is the external source, \( A(\vec{x}) \) and \( B(\vec{x}) \) are the space dependent Jacobian matrices at the \( \vec{x} \), \( \lambda, \mu \) are the Lame constants and \( \rho \) is the mass density of the material.

Zhang and Shen (2006) proposed ADE CFS-PML. Its core idea is to use the auxiliary differential equation to update the auxiliary variables of CFS-PML, which can be written in matrix form

\[
\begin{align*}
\frac{\partial \mathbf{u}}{\partial t} &= \frac{1}{\beta_x} A(\vec{x}) \frac{\partial \mathbf{u}}{\partial x} + \frac{1}{\beta_z} B(\vec{x}) \frac{\partial \mathbf{u}}{\partial z} + \frac{1}{\beta_x} A(\vec{x}) \mathbf{u}_x + \frac{1}{\beta_z} B(\vec{x}) \mathbf{u}_z, \\
\frac{\partial \mathbf{u}^i}{\partial t} + \left( \alpha_i + \frac{d_i}{\beta_i} \right) \mathbf{u}^i &= - \frac{d_i}{\beta_i} \frac{\partial \mathbf{u}}{\partial t}, (i = x, z)
\end{align*}
\]
where \( \vec{u}^x \) and \( \vec{u}^z \) are the auxiliary variables in PML layer, \( \alpha_x \) and \( \alpha_z \) are the frequency-shifted factors, \( d_x \) and \( d_z \) are the attenuation factors that cause the amplitude of the wavefield to be reduced exponentially inside the PML layer, \( \beta_x \) and \( \beta_z \) are the scaling factors.

We remove the source term for Equation (1) and perform a Fourier transform on it to obtain
\[
i\omega \vec{u} = A(\vec{x}) \frac{\partial}{\partial x} \left( \frac{\vec{u}}{s_x} \right) + B(\vec{x}) \frac{\partial}{\partial z} \left( \frac{\vec{u}}{s_z} \right),
\]
(3)
where \( s_x \) and \( s_z \) are the complex stretching functions that determine the characteristics of the PML model. The specific expressions of \( s_x \) and \( s_z \) are as follows
\[
s_x(\vec{x}) = \beta_x(\vec{x}) + \frac{d_x(\vec{x})}{\alpha_x(\vec{x}) + i\omega}, \quad s_z(\vec{x}) = \beta_z(\vec{x}) + \frac{d_z(\vec{x})}{\alpha_z(\vec{x}) + i\omega}.
\]
(4)
Thus,
\[
\frac{1}{s_i} = \frac{1}{\beta_i} = \frac{1}{\beta_i} - \frac{1}{\beta_i} \left( \frac{\alpha_i + i\omega}{\beta_i} \right) + \frac{d_i}{\beta_i + \alpha_i + i\omega}, \quad (i = x, z)
\]
(5)
By defining \( \Theta^i = -\frac{d_i}{\beta_i \left( \alpha_i + i\omega \right) \beta_i + d_i}, \quad (i = x, z) \), we can write Equation (4) as follows
\[
i\omega \vec{u} = A(\vec{x}) \frac{1}{\beta_x} \frac{\partial}{\partial x} (\vec{u} + \Theta^z \vec{u}) + B(\vec{x}) \frac{1}{\beta_z} \frac{\partial}{\partial z} (\vec{u} + \Theta^x \vec{u}),
\]
(6)
Introducing auxiliary variables \( \vec{u}^i = \Theta^i \vec{u}, \quad (i = x, z) \), we can get
\[
i\omega \vec{u}^i + (\alpha_i + d_i/\beta_i) \vec{u}^i = -\frac{d_i}{k_i} \vec{u}, \quad (i = x, z),
\]
(7)
Taking the inverse Fourier transform of equation (7) and (8), we get the improved ADE CFS-PML is
\[
\begin{align*}
\frac{\partial \vec{u}}{\partial t} &= \frac{1}{\beta_x} A(\vec{x}) \frac{\partial \vec{u}}{\partial x} + \frac{1}{\beta_x} B(\vec{x}) \frac{\partial \vec{u}}{\partial z} + \frac{1}{\beta_z} A(\vec{x}) \frac{\partial \vec{u}^x}{\partial x} + \frac{1}{\beta_z} B(\vec{x}) \frac{\partial \vec{u}^z}{\partial z}, \\
\frac{\partial \vec{u}^x}{\partial t} + (\alpha_x + \frac{d_x}{\beta_x}) \vec{u}^x &= -\frac{d_x}{\beta_x} \vec{u}, \\
\frac{\partial \vec{u}^z}{\partial t} + (\alpha_z + \frac{d_z}{\beta_z}) \vec{u}^z &= -\frac{d_z}{\beta_z} \vec{u},
\end{align*}
\]
(8)
We use DG-FEM proposed by Käser and Dumbser (2006) to solve equation (1) and (9), and the semi-discrete form can be achieved. More details can be found in Käser and Dumbser’s work. The time discretization, we use the strong stability-preserving Runge-Kutta (SSP-RK), as they guarantee that no additional oscillations are introduced as part of the time-integration process. We choose the fourth-order scheme by allowing a fifth stage, because the additional work is partially offset by the maximum timestep being approximately 50% larger than for the forward Euler method (Spiteri et al., 2002).

**Examples**

In order to verify the effect of the modified ADE CFS-PML, we conduct some simulation tests on the propagation of elastic waves in isotropic media.

First, we test the propagation of seismic waves in a full-space homogeneous medium model. The model size is 1km \( \times \) 1km, and the thickness of PML layer is set to be 200m (Figure 1 (a)). We use a triangular mesh to divide the model, and the side length of the mesh is about 10m (Figure 1 (b)). P-wave and S-wave velocity is 3000m/s and 2000m/s respectively, and density is 2000kg/m³. The explosive source (pentacle) is placed at (250m, 250m), and the receiver located at (250m,750m). We
choose 50hz Ricker wavelet as the source, and simulate wave propagation of 1s. The comparison is shown in Figure 2. It can be seen that modified ADE CFS-PML has a good absorption effect.

![Figure 1](image1.png)

*Figure 1* (a) schematic diagram of the model, (b) meshing model results.

![Figure 2](image2.png)

*Figure 2* (a) the results of open boundary condition, (b) the result of modified ADE CFS-PML, (c) compare Absorbing boundary condition and modified ADE CFS-PML seismograms.

Then, we test the propagation of seismic waves in a half-space model with free surface. The source is located at (1000m, 700m) and the receiver is located at (1000m, 100m), P-wave and S-wave velocity is 3000m/s and 1200m/s respectively, and density is 1800kg/m$^3$. Figure 3 shows the comparison of simulation results for different degrees of freedom and it indicates that the 2$^{nd}$-order of DG-FEM can perfectly match the analytic solution.
Figure 3 (a) (b) and (c) (d) are snapshots of wavefield of $V_z$ component at different moments with 1st and 2nd-order degree of freedom respectively; (e) is the comparison of GRTM analytical solutions with 1st and 2nd-order respectively; (f) comparison of 1st and 2nd-order seismograms.

Conclusions

In this paper, we study the application of ADE CFS-PML in DG-FEM. By introducing different auxiliary variables, we get a more concise equation set, which is successfully applied in DG-FEM. The comparison between numerical simulation and analytical solutions proves the reliability of our method. Further more, the modified PML is also suitable for higher-order time integration algorithms.

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References


