Introduction

The theory of poro-elasticity was established by Biot (1941), in the context of soil compaction. As end points of the dynamic process of soil compaction, he defined the "instantaneous compressibility" (characterizing the compressibility at the instant of application of a compressive load), and the "final compressibility". These are now known respectively as the "undrained" and "drained" compressibilities. In the drained condition, the external pressure is supported only by the frame.

Ten years after Biot's foundational work, Gassmann (1951) produced another analysis (not referencing Biot), relating the undrained and frame incompressibilities:

\[
K_{ud} = K_f + \frac{(1-K_f)K_s^2}{\phi(K_f^{-1} - K_s^{-1}) + (1-K_f)K_s^2K_f^{-1}}
\]

(1)

where the subscripts indicate respectively the incompressibility \(K\) of the undrained (\(ud\)) rock, the frame (\(f\)) of the rock, the Solid (\(S\)) grains of the rock, and the Fluid (\(F\)) in the pore space of the rock; \(\phi\) is porosity. This result (1) has been used extensively ever since, for example in exploration geophysics, to help understand the fluid content of subsurface rocks. It is frequently called the "Biot-Gassmann formula", but this is a misnomer, since it has fewer parameters (fewer degrees of freedom) than does the corresponding result from Biot; the two results are mutually inconsistent.

In the present analysis, first the Biot result, and then the Gassmann result are re-cast exactly so that the inconsistency is readily apparent. Then, the difference is traced to an erroneous application by Gassmann of an open-system theorem to the closed-system of low-frequency wave propagation. Insights regarding this error are provided by the subsequent analysis of Brown and Korrinng (1975) ("B&K"). Finally, a feasible means for determining the additional parameter is presented. The argument is presented with detailed references to the original papers, so that readers may verify it, for themselves, even after the conference in 2020.

Biot

Biot (1941) considered a sample of soil (or of rock), macro-isotropic and statistically uniform, either micro-homogeneous or not, either micro-isotropic or not. The macro-isotropic constitutive equation for isotropic compression is given, in Biot's notation, by the sum of his equations (2.4) for three components of normal strain \(e_i\) (yielding the dilatation \(\delta V/V\)):

\[
e_x + e_y + e_z = \frac{\partial V}{V} = \frac{(1-2\nu)(\sigma_x + \sigma_y + \sigma_z)}{E} + \frac{\sigma}{H} = \frac{(1-2\nu)(-3p)}{E} + \frac{\sigma}{H} = -\kappa_f p + \frac{\sigma}{H}
\]

(2)

where \(V\) is the specific volume of rock, and \(\delta\) indicates a change due to compression. The \(\sigma_i = p\) are normal components of incremental confining pressure, and \(\sigma\) is the incremental fluid pressure, assumed to be uniform. The classical elastic parameters of what Biot called the "skeleton" (in modern usage: the "frame") of the rock are its Poisson's ratio \(\nu\) and its Young's modulus \(E\), replaced by the frame compressibility \(K_f = 1/K_p\) in the final form above. \(H\) is a poro-elastic parameter (not appearing in classical elasticity) first defined by Biot (1941).

Biot discussed the "instantaneous compressibility \(a_i\)" in the context of uniaxial strain at his equation (3.10); here the same concept is applied to isotropic compression and strain. Before any fluid leaves the sample (but after any pore-scale inhomogeneities of fluid pressure have relaxed), the 3D instantaneous compression is given, from equation (2), by the undrained compressibility \(K_{ud} = 1/K_{ud}\).

\[
\frac{\partial V}{V} = -\kappa_{ud} p = -\kappa_f p + \frac{\sigma}{H}
\]

(3)

Similarly, Biot gave (his equation (2.6)) the "increment of water content \(\theta\)" as

\[
\theta = \frac{1}{3H_1} \left( \sigma_x + \sigma_y + \sigma_z \right) + \frac{\sigma}{R}
\]

(4)

where \(R\) is a second (independent) poro-elastic parameter. It was subsequently proven by Biot (using an energy argument, p. 158) that \(H_1 = H\). For undrained isotropic (3D) compression, the increment \(\delta V_F\) of specific volume of fluid is given by the fluid compressibility \(K_f = 1/K_f\):
\[
\theta = \frac{\delta V}{V} = \frac{V}{V} \left( \frac{\delta V}{V} \right) = \phi (-\sigma F) = -\frac{p}{H} + \frac{\sigma}{R}
\]  
(5)

Eliminating the ratio \( \sigma/p \) from equations (3, 5) yields the relationship between undrained and frame compressibilities:

\[
\kappa_{ud} = \kappa_{fr} - \frac{H^{-2}}{\kappa_{fr} + R^{-1}}
\]  
(6)

showing explicitly the two poro-elastic parameters, \( H \) and \( R \). Although (6) was not stated explicitly by Biot (1941), it follows so directly from his foundational work that it is here attributed to him.

**Gassmann**

Gassmann's (1951) result for Biot's case (but assuming micro-homogeneity of the solid), for the undrained compressibility is given in his §59, in unfamiliar notation. (Note that, in the English translation, there is a typographical error.) This result of Gassmann is quoted here in the more familiar notation of (1). It is easy to re-cast (1) exactly in terms of compressibilities \( \kappa = 1/K \) with the same meanings of the subscripts:

\[
\kappa_{ud} = \kappa_{fr} - \frac{(\kappa_{fr} - \kappa_S)^2}{(\kappa_F - \kappa_S) \phi + \kappa_{fr} - \kappa_S}
\]  
(7)

In this form, Gassmann's result (7) may be directly compared to Biot's result (6).

**Gassmann vs Biot**

Comparing equations (6) and (7), it is easy to make the correspondence in notation:

\[
H^{-1} \rightarrow \kappa_{fr} - \kappa_S; \quad R^{-1} \rightarrow -\phi \kappa_S + \kappa_F - \kappa_S = -\phi \kappa_S + H^{-1}
\]  
(8)

It is clear that the two poro-elastic parameters (\( H \) and \( R \)) of Biot were replaced by Gassmann with one parameter (\( \kappa_S \)). Although Gassmann's result is restricted to the case of solid micro-homogeneity, this restriction is not responsible for the difference, which arises instead from a logical error made by Gassmann, discussed below. Gassmann's derivation is logically divided into sections:

- preliminaries §§1-42,
- consideration of hydraulically open systems, §§43-52,
- consideration of hydraulically closed systems, §§53-64,
- a numerical example, §§64-74, and anisotropy §§75-94.

The cartoon shows uniaxial compression in an open system (on the left) with pore fluid ejected or injected from an external reservoir, and in a closed system (on the right) where, as in wave propagation, no fluid enters or leaves.

As indicated in the cartoon above, in "open" systems, fluid is exchanged with an external reservoir. In Gassmann's section on open systems, he correctly derived (in §45) a result which may be expressed as
Here, the subscript unjacketed indicates the hydraulically open condition wherein the increment in confining pressure $p$ is accompanied by an equal increment in fluid pressure $\sigma$. This theorem was first derived by Love (1927, §§121, 123(iii, iv)), it has been used by many analysts over the decades since. It is strictly valid for a micro-homogeneous solid, and approximately true for a micro-heterogeneous solid.

But, in both cases, it is valid only for the unjacketed (hydraulically open) systems where it was derived. (In order to achieve the unjacketed increased pressure state, additional fluid must be injected into the sample from an external reservoir.) Hence, the result (9) cannot be applied in a closed (undrained) context, such as Gassmann’s subsequent section, beginning at §53.

Nevertheless, Gassmann did apply it at §58d (in his closed-system section), in his derivation of the present equation (7); this step produced the inconsistency with (6). This logical error by Gassmann means that his final result (1) is in error, and should be replaced by Biot’s prior result (6). This causes serious difficulties in application, since Biot did not indicate how to determine $H$ and $R$, and since the solid compressibility $k_s$ does not appear in (6) at all. Insight on these issues in provided by B&K.

**Brown and Korringa ("B&K")**

B&K’s derivation (more thorough than Biot’s) produced an expression (their equation (2)) which is exactly equivalent to (6). In their notation:

$$k^* = k_A - (k_A - k_M)^2 / (\phi (k_F - k_{\phi}) + (k_A - k_M))$$  

(This B&K result was reproduced, with different notation, by Rice and Cleary (1976).) In (10), the correspondence to the notation of (6) is:

$$k_{ud} \rightarrow k^*; \quad H^{-1} \rightarrow k_A - k_M$$

$$k_{fr} \rightarrow k_A; \quad R^{-1} \rightarrow k_{\phi}^* + k_A - k_M = H^{-1} - k_{\phi}^*$$

In B&K’s result (10), the strange parameters $k_{ud}$, $k_M$, $k_{\phi}^*$ were defined by B&K at their equations (4abc), respectively. B&K describe $k_{\phi}$ as the “pore compressibility”, but do not indicate the mnemonics behind the other two subscripts, $A$ and $M$. Taking B&K’s title seriously (check the **References**, below!), equation (10) shows that in the asymptotic limit of large $k_{\phi}$, the undrained compressibility becomes $k^*(k_{\phi} \rightarrow \infty) = k_{ud}$ ($k_{\phi} \rightarrow \infty) = k_A$. Hence, $k_A$ is here identified as the “Asymptotic compressibility”.

Thus, $k_A = k_{fr}$ is defined through the functional dependence of $k_{ud}$ upon $k_{fr}$, rather than through (open-system) drainage of the rock. In fact, B&K did not consider open hydraulic systems at all; they dealt only with the closed system (cf. their equation (1)), and its dependence on pore fluid compressibility. This dependence of $k_{ud}$ upon $k_{fr}$, might, in principle, be determined experimentally through fluid substitution (e.g. to a gas, with very high $k_{fr}$). However, $k_A = k_{fr}$ is also numerically equal to the drained compressibility $k_{fr}$, since in neither case (extreme fluid substitution nor drainage) does the fluid support the compressional load; operationally, this is easier to accomplish. But, $k_A$ is defined in terms of the functional dependence of $k_{ud}$ upon $k_{fr}$.

B&K also provided a relation between the other strange parameter, $k_M$, and the missing parameter $k_{\phi}$, through an unnumbered equation on p. 614:

$$k_M = (1 - \phi) k'' + \phi k_{\phi} = (1 - \phi) k_S + \phi k_{\phi}$$

where B&K’s parameter $k''$ is here recognized as the average solid compressibility $k_S$ (cf. their discussion on p. 614). Because of the functional form of (12), $k_M$ is here identified as the "Mean compressibility" (not the mineral compressibility, which is $k_S$).
B&K argued (p. 610) that, when their result (10) is specialized to Gassmann’s case of microhomogeneous rocks, \( \kappa_f = \kappa_p = \kappa_s \), hence recovering Gassmann’s result (7). Because of this B&K argument, it has been widely understood that the additional term in B&K’s result (10) is due to microheterogeneity (and further, that this is a minor issue, since the common minerals are broadly similar).

However, in that argument (p. 610), B&K made the same logical error that Gassmann did (applying Love’s open-systems theorem to a closed system). Their reduction to Gassmann’s result came from that logical error, not from the restriction to micro-homogeneous solid (since Love’s theorem (9) is approximately valid, even in the micro-heterogeneous case, especially accurate since the common minerals are broadly similar in elasticity). Hence B&K’s result (10) (or Biot’s result (6)) for the undrained compressibility is valid as written above (with \( \kappa_f \) and \( \kappa_p \) independent, or with \( H \) and \( R \) independent) for all macro-isotropic, statistically uniform rocks, with uniform pore pressure.

Using (12) and (11) to eliminate \( \kappa_S \) from equation (10) in favor of \( \kappa_S \):

\[
\kappa_{ud} = \kappa_f - \frac{\left( \kappa_f - \kappa_M \right)^2}{\phi(\kappa_f - \kappa_S) + \kappa_f + \kappa_S - 2\kappa_M} = \kappa_f - \frac{H^{-2}}{\phi(\kappa_f - \kappa_S) + 2H^{-1} - (\kappa_f - \kappa_S)}
\]

Hence, it is not necessary to determine the pore compressibility \( \kappa_p \).

In Equation (13), Biot’s parameter \( H \) (or B&K’s parameter \( \kappa_M \)) may be determined, from equation (2):

\[
H = \left( \kappa_f - \kappa_M \right)^{-1} = \frac{\left( \sigma / p \right)}{(\kappa_f - \kappa_M)} = \frac{B}{(\kappa_f - \kappa_M)}
\]

where the ratio \( \sigma / p \) of undrained fluid pressure to confining pressure (“Skempton’s \( B \)-coefficient”) is directly observable, with a quasi-static experiment, even with low imposed strains.

**Discussion**

It is remarkable to identify a logical error in a theory which has been well-accepted for 2/3 of a century, including 15 years since the English translation appeared. However, the logic is simple and clear, and the implications are serious. The logical error in the derivation of Gassmann’s equation (1) means that the result of Biot (equations (6, 13)), or of B&K (equations (10, 13)) must be used instead to analyze the fluid content of rocks. The additional parameter \( H \) or \( \kappa_M \) may be determined via (14), using low-frequency (quasi-static) data.

Such low-frequency tests can determine the validity (or not) of the common result of Biot (1941), and of Brown and Korringa (1975). This determination, performed on many rock samples, under many pressure conditions, is necessary in order to establish the practical significance of the logical error described herein. If \( \kappa_M \) commonly turns out to be significantly different from \( \kappa_S \), this would imply that the practical significance is usually large. In this case, it may also be economically significant, e.g. for interpreting 4D changes in reservoir fluids and pressures from 4D seismic data. Determining such rock physics systematics is an important task for the future, for rock physics experimentalists.

**References**


