An approximation for calculating R/T coefficients in viscoelastic VTI media

Introduction

Viscoelastic and anisotropic properties of subsurface rocks have a profound influence on seismic wave propagation confirmed by laboratory measurements (Clark et al., 2009). When the amplitude attenuation and phase of a seismic wave vary with direction of propagation in a given medium, we often term the medium a viscoelastic anisotropic medium (VEAM). The wave propagation in such a medium is characterized by the slowness vector, phase speed, and ray velocity of the wave, which are all complex-valued and frequency-dependent quantities, and defined by the complex elastic moduli of the medium. The real parts of these quantities specify the kinematic wave propagation along a raypath, while the imaginary parts describe the decay of wave amplitude (or energy) due to dissipation of seismic waves (Carcione, 2007). In order to obtain the exact solutions of the seismic raypath in a VEAM, a complex ray tracing method is required and very challenging due to the high complexity of computation in the complex domain.

To simplify the complex ray tracing and improve its computational efficiency, many approximations are proposed, such as Newton inversion scheme, solving algebraic equations of the sixth degree, real ray tracing method (RRT) (Vavryčuk, 2008), and perturbation methods (Vavryčuk, 2008; Wu et al., 2020). These approaches are all applicable to the smoothly inhomogeneous VEAM. For a layered VEAM model, the reflection and transmission (R/T) of seismic wave must be considered, so that the seismic waves follow the complex Snell’s law. The computation of the R/T coefficients is also a changeling task. Vavryčuk (2010) employed the real Snell’s law in the RRT method and demonstrated an iterative scheme to obtain the R/T coefficients. However, the iterative approach has three disadvantages: (1) it cannot deal with the triplication of the qSV wavefronts that is often encountered in anisotropic media, (2) it is limited to the range of pre-incident critical angles due to the stationary condition of homogeneous ray velocity vector, and (3) it is inefficient and time-consuming for a complex VEAM. In this paper, we propose a new approximate method to calculate the R/T coefficients of seismic wave in viscoelastic transversely isotopic media with a vertical axis of symmetry (VTI), which can deal with the limitations above and also be extended into a generalized VEAM.

The real slowness direction (RSD) approximation

Recently, Wu et al (2020) proposed the RSD approximation for the phase slowness vector in a VEAM,

$$p = n / \nu = \frac{\nu^*}{\nu} n,$$

where $n$ is approximated by $n \approx (\cos \theta R^v, 0, \sin \theta R^v)$ with a real angle $\theta R^v$ and $\nu^*$ are the complex and its conjugate phase velocity defined by the eigenvalue of the Christoffel equation:

$$\det\left( a_{ij} n_j n_i - \nu^2 \delta_{ik} \right) = 0.$$

Here, $a_{ij}$ is the fourth-order tensor of the complex density-normalized moduli. As the wave propagate across the interface, the real Snell’s law must be satisfied

$$p_{i|\text{in}} = p_{o|\text{out}},$$

where the subscripts ‘in’ and ‘out’ stand for the incident and reflected or transmitted waves. Eq. 3 gives the relationship of the tangential component $p_x$ of the slowness vector between the ‘in’ and ‘out’ waves. The link to the vertical component $p_z$ is also required for the complete slowness vector $\vec{p} = (p_x, p_z)$. Fortunately, the Hamiltonian function may be employed to determine the vertical components, e.g. for the qP-, qSV- and qSH-waves in a viscoelastic VTI media, we have

$$H (\vec{x}, \vec{p}) = (a_{44} + a_{11}) (p_z^2)^2 + (a_{13} + a_{44}) p_z^2 - a_{44} a_{44} p_z^2 - a_{44} a_{44} p_z^2 - Dp_z^2 p_z^2 - 1 = 0$$

$$D = a_{33} (a_{13} - a_{33} + 2a_{44}) - (a_{13} + a_{44})^2 + (a_{33} - a_{44})^2,$$  

$$H (\vec{x}, \vec{p}) = a_{44} p_z^2 + a_{44} p_z^2 - 1 = 0, \quad (qP- & qSV-wave).$$

$$H (\vec{x}, \vec{p}) = a_{44} p_z^2 + a_{44} p_z^2 - 1 = 0, \quad (qSH-wave).$$

According to eqs. 4 and 5, one can calculate the vertical component $p_z$ with the known moduli $a_{ij}$ and the tangential component $p_x$, and then obtain the slowness vector $\vec{p} = (p_x, p_z)$ with the RSD
approximation. In general, the slowness vector \( \vec{p} \) is an inhomogeneous complex vector, for which a complex ray tracing scheme is naturally applied. Unfortunately, none of the complex ray tracing methods can generate the R/T solutions for all the three wave modes (qP, qSV, qSH) in a VEAM, while eqs.1–5 imply that the RSD approximation may replace the RRT method to efficiently calculate the slowness angle with the slowness components \( (p_x, p_y) \) that satisfy eq. 4 or 5,
\[
\theta = \arctan \left( \frac{p_y}{p_x} \right).
\]
Therefore, the ray velocities, ray attenuations and polarization directions of the three wave modes in a VEAM can be calculated by the RSD method (Wu et al. 2020).

To compute the R/T coefficients of seismic waves, the boundary condition of the continuity of displacement vector \( \mathbf{u} \) and traction vector \( \mathbf{T} \) on the interface must be satisfied (Vavryčuk, 2010). We write the seismic waves in the plan-wave forms,
\[
\mathbf{u}^{(w)}(\mathbf{x}, \omega) = c^{(w)} \mathbf{g}^{(w)} \exp \left[ i \omega \tau^{(w)}(\mathbf{x}) \right],
\]
\[
\mathbf{T}^{(w)}(\mathbf{x}, \omega) = c^{(w)} \mathbf{\sigma}^{(w)} \exp \left[ i \omega \tau^{(w)}(\mathbf{x}) \right],
\]
where \( \tau \) and \( c \) are the traveltime and amplitude, the superscript \( W=1 \)–6 represent the reflected and transmitted waves of the three wave modes (qP, qSV, qSH). The vector \( \mathbf{\sigma}^{(w)} \) is the amplitude-normalized traction vector, whose components are given by
\[
\sigma_{ij}^{(w)} = \rho^{(w)} g_{ij}^{(w)} g_{kl}^{(w)} p_k^{(w)}. \tag{9}
\]
Here, \( \rho \) is the density and \( \mathbf{g} \) is the polarization of the wave. The boundary condition means
\[
\mathbf{u}^{(0)}(\mathbf{x}, \omega) = \sum_{W=1}^{6} \mathbf{u}^{(w)}(\mathbf{x}, \omega), \tag{10}
\]
\[
\mathbf{T}^{(0)}(\mathbf{x}, \omega) = \sum_{W=1}^{6} \mathbf{T}^{(w)}(\mathbf{x}, \omega).
\]
Substituting Eqs. 8 and 9 for (10), we obtain the following equations for the reflection/transmission:
\[
\mathbf{Dc} = \mathbf{a}^{(0)} \text{ or } \mathbf{c} = \mathbf{D}^{-1} \mathbf{d}^{(0)}. \tag{11}
\]
Here, \( \mathbf{c} \) is the vector of either the R/T coefficients for the qP- and qSV-wave or the qSH-wave. \( \mathbf{D} \) is a \( 4 \times 4 \) or \( 2 \times 2 \) displacement-stress matrix that corresponds to the qP-, qSV- and qSH-waves, respectively.

Solving eq. 11 with the RSD approximation (eq.1), we obtain the R/T coefficients for the reflected and transmitted qP-, qSV-, and qSH-waves in viscoelastic VTI media. Comparing the RSD method with the RRT method proposed by Vavryčuk (2010), one may find that the former may directly compute the solutions and the latter requires an iterative search for the complex slowness direction and is only applicable the qSH-wave solutions. To examine the qP- and qSV-wave solutions of the RSD approximation, we employ the identity \( \mathbf{gg}^* = 1 \) to normalize the eigenvector \( \mathbf{g} \) in the RRT method, which is named the generalized RRT method (GRRT) that enables to deal with the qP- and qSV-waves, so that the comparison between the RSD approximation and the GRRT solution is implemented.

**NUMERICAL EXAMPLES**

To examine the accuracy and efficiency of the RSD approximation, we use the viscoelastic parameters of two VTI layers (A and B) given in Table 1, which have the properties of strong anisotropy and attenuation.

The first experiment was conducted for the reflected and transmitted qP- and qSV-wave with the incident qP-wave for the pre-critical angles in the two-layered model. Figure 1 shows the reflected and transmitted ray angles and ray velocities of the qP- and qSV-waves, as well as the errors compared with the GRRT solutions. These results indicate that the approximate results for the qP- and qSV-waves show high accuracy of their ray angles and ray velocities for all the incidents angles. Figure 2 shows the computed R/T coefficients generated by the RSD approximation and GRRT method for the qP- and qSV-waves. These results exhibits that although the computed phases may be significantly different, the R/T coefficients of the qP- and qSV-wave yielded by the RSD approximation match well with the GRRT solutions, and the results of qP-wave display higher accuracy than that of the qSV-wave.
The 2nd experiment was conducted with the model having the same parameters given in Table 1, but we exchange the order of the two layers, so that the overcritical incidences will occur in this case. Figure 3 shows the computed ray angles and ray velocities for the qP- and qSV-waves. Figure 4 displays the computed modulus and phases of the R/T coefficients for qP- and qSV-waves. From these results, one may find that the critical angle of the incident qP-wave is about 57° and the GRRT method fails in generating the solutions of the R/T coefficients for both qP- and qSV-waves with any post-critical incident angle (see black lines in Fig. 3 and 4), but the RSD approximation does offer the approximate solutions of the R/T coefficients (see red dashed lines in Fig. 3 and 4).

Table 1. The viscoelastic parameters of two VTI layers

<table>
<thead>
<tr>
<th>Model</th>
<th>Elastic parameters (km^2/s^2)</th>
<th>Attenuation parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(a_{11}^E)</td>
<td>(a_{13}^E)</td>
</tr>
<tr>
<td>A</td>
<td>14.4</td>
<td>4.5</td>
</tr>
<tr>
<td>B</td>
<td>9.0</td>
<td>4.0</td>
</tr>
</tbody>
</table>

Figure 1 The computed ray angles and ray velocities of the reflected and transmitted qP-waves (\(R_{qP}, T_{qP}\)) and qSV-waves (\(R_{qSV}, T_{qSV}\)), and their errors compared with the GRRT solutions for the model given in Table 1.

Figure 2. The computed modulus and phase of the reflected and transmitted qP- and qSV-waves for the model given in Table 1.

Figure 3 The computed ray angles and velocities of the reflected and transmitted qP-waves (\(R_{qP}, T_{qP}\)) and qSV-waves (\(R_{qSV}, T_{qSV}\)), and their errors compared with the GRRT solutions for the two-layered model, in which the 1st and 2nd layers become B and A given in Table 1, respectively.
**Fiuger 4.** The computed modulus and phase of the reflected and transmitted qP- and qSV-waves for the model, in which the 1st and 2nd layers are B and A given in Table 1, respectively.

**Conclusions**

We propose an approximate method that can be employed to effectively and efficiently calculate the R/T coefficients in heterogeneous viscoelastic VTI media. Two numerical examples demonstrate that the approximate method may yield satisfactory solutions of the R/T coefficients of the qP- and qSV-wave for the incident qP-wave within the pre- and post-critical angles, and is also applicable for the incident qSV- and qSH-waves (not shown here).

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**Reference**


