Introduction

Anisotropy plays a vital role in the seismic wave simulation. In the transversely isotropic (TI) media, the P- and SV-wave phase velocity expressions with multiple anisotropic parameters are complex. In order to simplify the P-wave phase velocity expression in TI media, Alkhalifah (1998) proposed the acoustic approximation for P-wave by setting the vertical S-wave velocity to zero. Based on this theory, many pseudo-acoustic wave simulation methods in the vertically transversely isotropic (VTI) media and tilted transversely isotropic (TTI) media were proposed in the past two decades.

The coupled equations solved by the finite difference (FD) method were proposed to solve the 4th-order pseudo-acoustic wave equation (Alkhalifah, 2000; Zhou et al., 2006). In the resulting wavefields, the P-wave and instable SV-wave artifacts are coupled (Grechka et al., 2004; Jin and Stovas, 2018). Then, the pseudo-spectral (PS) method was introduced to eliminate the SV-wave artifacts (Pestana and Stoffa, 2010; Zhan et al., 2013). However, these methods are only valid on the assumption of the weak anisotropy approximation with lower accuracy. For the complex 2nd-order pure P-wave equation of the acoustic approximation, the low-rank approximation can be utilized to calculate the complicated mixed-domain operator. But in the complex TTI models, multiple Fourier transforms at each time step leads to expensive computing cost (Fomel et al., 2013).

In this abstract, we propose an optimized acoustic approximation in TI media and the corresponding pure P-wave simulation method in TTI media to cover the shortages of the standard acoustic approximation and its applications. First, we construct a function of the vertical S-wave velocity squared to obtain an approximate P-wave phase velocity expression in TI media with a concise format. Then, we derive the corresponding 2nd-order pure P-wave equation in TTI media. In order to demonstrate the advantages, we compare the wavefields of two previous approaches (Zhou et al., 2006; Zhan et al., 2013) and the proposed method in homogeneous and complex TTI models.

Theory

We start from the exact P- and SV-wave phase velocity expressions in TI media (Thomsen, 1986):

\[ v_{P}^2(\theta) = 0.5[(1 + 2\varepsilon \sin^2 \theta)v_{p0}^2 + v_{S0}^2 + D], \]

\[ v_{SV}^2(\theta) = 0.5[(1 + 2\varepsilon \sin^2 \theta)v_{p0}^2 + v_{S0}^2 - D], \]

\[ D = \sqrt{(1 + 2\varepsilon \sin^2 \theta)v_{p0}^2 - v_{S0}^2} + 2 \varepsilon \sin^2 \theta \cos^2 \theta (v_{p0}^2 - v_{S0}^2 v_{S0}), \]

where \( \theta \) is the phase angle, \( \varepsilon \) and \( \delta \) are Thomsen’s parameters, \( v_{p0} \) and \( v_{S0} \) are vertical P- and S-wave velocities, respectively. By setting \( v_{S0} \) to zero, the P-wave phase velocity expression of the standard acoustic approximation is given by (Alkhalifah, 1998)

\[ v_{p0}^2(\theta) = 0.5v_{p0}^2\left[1 + 2\varepsilon \sin^2 \theta + \sqrt{(1 + 2\varepsilon \sin^2 \theta)^2 - 2(\varepsilon - \delta) \sin^2 \theta} \right]. \]

P-wave propagation in anisotropic media is weakly dependent on \( v_{S0} \). This fact gives us the reason to manipulate \( v_{S0} \) in order to get rid of SV-wave and keep sufficient accuracy for P-wave. For the standard acoustic approximation, the square root term in equation (4) is a barrier for its application in pure P-wave extrapolation. So, rather than setting \( v_{S0} \) to zero, our goal is to construct a function of \( v_{S0}^2 \), which is dependent on the other parameters, to derive a more concise approximate P-wave phase velocity expression. Based on equation (2), this function of \( v_{S0}^2 \) can be represented as

\[ v_{S0}^2(\theta) = 2v_{p0}^2(\theta) + D - (1 + 2\varepsilon \sin^2 \theta)v_{p0}^2. \]

According to equation (1), (3) and (5), the corresponding P-wave phase velocity expression is

\[ v_{P}^2(\theta) = \frac{1 + 2\varepsilon \sin^2 \theta - \frac{2(\varepsilon - \delta) \sin^2 \theta \cos^2 \theta (1 - v_{S0}^2(\theta)/v_{p0}^2)}{1 + 2\varepsilon \sin^2 \theta + 2\delta \sin^2 \theta \cos^2 \theta - v_{S0}^2(\theta)/v_{p0}^2}}{v_{p0}^2}. \]
where

\[ v^s_{20}(\theta) = v^s_{20}(2\sin^2 \theta + 2\delta \sin^2 \theta \cos^2 \theta). \]  

(7)

Substituting equation (7) into equation (5), we receive the function of \( v^2_{20} \) as

\[ v^s_{20}(\theta) = v^s_{20}[2\sin^2 \theta - 4(\epsilon - \delta) \sin^2 \theta \cos^2 \theta(1 - \epsilon \sin^4 \theta - \delta \sin^2 \theta \cos^2 \theta)]. \]  

(8)

Then, we derive an approximate P-wave phase velocity expression in TI media:

\[ \omega^2 = v^2_{20} \left[ (1 + 2\epsilon) k_i^2 + k_r^2 - 2(\epsilon - \delta)(k_i^2 + k_r^2) \right]\left[k_i^2 k_r^2 (k_i^2 + k_r^2)^2 - 4\epsilon k_i^2 k_r^2 (k_i^2 + k_r^2)^2 \right]. \]  

(9)

Substituting equation (8) into equation (5), we receive the function of \( v^2_{20} \) as

\[ \omega^2 = v^2_{20} \left[ (1 + 2\epsilon) k_i^2 + k_r^2 - 2(\epsilon - \delta)(k_i^2 + k_r^2) \right]\left[k_i^2 k_r^2 (k_i^2 + k_r^2)^2 - 4\epsilon k_i^2 k_r^2 (k_i^2 + k_r^2)^2 \right]. \]  

(10)

After rotating the wavenumbers as

\[ \hat{k}_i = \cos \phi k_i + \sin \phi k_r, \quad \hat{k}_r = -\sin \phi k_i + \cos \phi k_r, \]  

(11)

where \( \phi \) is the dip angle of the symmetry axis, we obtain the dispersion relationship in TTI media as

\[ \omega^2 = v^2_{20} \left[ (1 + 2\epsilon) \cos^2 \phi k_i^2 + (1 + 2\epsilon \sin^2 \phi) k_r^2 + 2\epsilon \sin 2\phi k_i k_r - 2(\epsilon - \delta)(k_i^2 + k_r^2) \right] \sum_{i=1}^{3} \Gamma_i k_i^2 k_r^2. \]  

(12)

In equation (12), the auxiliary parameters \( \Gamma_{1-9} \) are computed as

\[ \Gamma_1 = 4\epsilon \sin(3\phi - 2\sin \phi) \cos^2 \phi - 0.25(2 - \delta + \cos 2\phi) \sin 4\phi \]  

(13)

\[ \Gamma_2 = 0.0625[8 - 5\epsilon - 3\delta - 2\epsilon \cos 2\phi + 4(\epsilon - \delta) \cos 4\phi - 14\epsilon \cos 6\phi - 7(\epsilon - \delta) \cos 8\phi] \]  

\[ \Gamma_3 = -0.125[3\epsilon \sin 2\phi + 2(\epsilon - \delta) \sin 4\phi + 7\epsilon \sin 6\phi + 7(\epsilon - \delta) \sin 8\phi] \]  

\[ \Gamma_4 = 0.03125[3(8 - 5\epsilon - 3\delta)] + 20(2 - \epsilon - \delta) \cos 4\phi + 35(\epsilon - \delta) \cos 8\phi] \]  

\[ \Gamma_5 = -0.125[3\epsilon \sin 2\phi - 2(\epsilon - \delta) \sin 4\phi + 7\epsilon \sin 6\phi - 7(\epsilon - \delta) \sin 8\phi] \]  

\[ \Gamma_6 = 0.0625[8 - 5\epsilon - 3\delta + 2\epsilon \cos 2\phi + 4(\epsilon - \delta) \cos 4\phi + 14\epsilon \cos 6\phi - 7(\epsilon - \delta) \cos 8\phi] \]  

\[ \Gamma_7 = -4\epsilon \sin(3\phi + 2\cos \phi) \sin^2 \phi - 0.25(2 - \delta + \cos 2\phi) \sin 4\phi \]  

\[ \Gamma_8 = (1 - 2\epsilon \sin^4 \phi - 2\delta \sin^2 \phi \cos^2 \phi) \sin^2 \phi \cos^2 \phi. \]  

According to the relationships: \( i\omega \leftrightarrow \partial / \partial t, \quad ik_x \leftrightarrow \partial / \partial x \) and \( ik_z \leftrightarrow \partial / \partial z \), we apply the inverse Fourier transform and multiply the pressure wavefield \( P \) to both sides of equation (12). Then, the pure P-wave equation in TTI media is derived as

\[ \frac{1}{v^2_{20}} \frac{\partial^2 P}{\partial t^2} = (1 + 2\epsilon \cos^2 \phi) \frac{\partial^2 P}{\partial x^2} + (1 + 2\epsilon \sin^2 \phi) \frac{\partial^2 P}{\partial z^2} + 2\epsilon \sin 2\phi \frac{\partial^2 P}{\partial x \partial z} \]  

\[ -2(\epsilon - \delta) F^{-1} \left[ \left( \frac{k_i^2}{k_i^2 + k_r^2} \right)^3 F \left( \Gamma_1 \frac{\partial^2 P}{\partial x^2} + \Gamma_2 \frac{\partial^2 P}{\partial x \partial z} + \Gamma_3 \frac{\partial^2 P}{\partial z^2} \right) \right], \]  

(14)

where \( F \) and \( F^{-1} \) represent the Fourier transform and inverse Fourier transform, respectively.

### Numerical experiments

In the following experiments, we choose the coupled equations by Zhou et al. (2006) and the hybrid FD/PS scheme by Zhan et al. (2013) to calculate the wavefields of the standard acoustic approximation. Due to the defects of the standard acoustic approximation, the hybrid FD/PS scheme is based on the weak anisotropy approximation with lower accuracy. And we use the proposed modelling method to calculate the wavefields of the optimized acoustic approximation.
Firstly, we design a homogeneous TTI model by the parameters of a practical shale sample. The vertical P-wave velocity is 3048 m/s. The Thomsen’s parameters $\varepsilon$ and $\delta$ are 0.255 and -0.05. We set the dip angle as 20°. The time sampling and the space sampling are 1 ms and 10 m, respectively. The source with the main frequency of 20 Hz is located in the middle of the model. The wavefield snapshots at 0.5 s are shown in Figure 1(a)-(c). The P-wave propagation of coupled equations suffers from the SV-wave artifacts. And the other two wavefields only involve P-wave propagation. The comparison of the slices at the depth of 2.0 km in three wavefields is shown in Figure 2(a). For the hybrid FD/PS method, the phase error of the P-wave is visible. In contrast, the optimized acoustic approximation is more accurate and effective for the pure P-wave extrapolation.

**Figure 1** The wavefields of the three methods in the homogeneous model: (a) Coupled equations, (b) Hybrid FD/PS scheme and (c) Proposed method.

**Figure 2** Comparison of the same slice in the three wavefields: (a) Homogeneous model and (b) Partial BP TTI benchmark model.

Then, we test the proposed method in the partial BP TTI benchmark model. The parameters are shown in Figure 3(a)-(d). The time sampling interval and the space sampling interval are 1 ms and 12.5 m, respectively. The source with the main frequency of 20 Hz is located in the middle of the model.

**Figure 3** The parameters of the partial BP TTI benchmark model: (a) Vertical P-wave velocity, (b) $\varepsilon$, (c) $\delta$ and (d) Dip angle.

The wavefield snapshots at 1.2 s are shown in Figure 4(a)-(c). Obviously, the wavefield of the coupled equations is unstable around the complex fault structure because of the SV-wave artifacts. And the other two wavefields are stable and free of SV-wave artifacts. The comparison of the slices at the depth of 6.5 km in the wavefields is shown in Figure 2(b). Although the hybrid FD/PS scheme can be used for stable
pure P-wave extrapolation, but the phase error of P-wave (green curve) is obvious due to the weak anisotropy approximation. By contrast, the wavefield of the proposed method only contains pure P-wave with high precision.

Figure 4 The wavefields of the three methods in the partial BP TTI benchmark model: (a) Coupled equations, (b) Hybrid FD/PS scheme and (c) Proposed method.

Conclusions

We propose an optimized acoustic approximation for P-wave in TI media and the corresponding pure P-wave extrapolation in TTI media. Since the P-wave phase velocity is weakly dependent on the vertical S-wave velocity, we derive an approximate P-wave phase velocity expression without square roots or complicated fractions by constructing a function of the vertical S-wave velocity squared. The 2nd-order pure P-wave equation in TTI media is derived directly from this 2nd-order expression without further low-accuracy simplifications. Solving it requires three Fourier transforms and one inverse Fourier transform at each time step, and this number will be further reduced in our following work. In the numerical experiments, the wavefields of the proposed method are completely free of instable SV-wave artifacts. Meanwhile, for the P-wave propagation, the accuracy of the standard and optimized acoustic approximations is basically the same. Therefore, the proposed method can be used for pure P-wave propagation in TTI media with high accuracy, and its computing cost is acceptable. It is valuable and practical for applying reverse time migration and full-waveform inversion in anisotropic media.

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References