Introduction

Several techniques in seismic data processing, e.g. migration or Full Waveform inversion, require the calculation of the wave field by numerically solving the wave equation several times over the acquisition domain. This makes these techniques computationally expensive (Virieux and Operto, 2009). For conventional techniques we need to solve the equation in all points of the domain delimited by seismic survey. However, there are some important cases where the overburden is known and the interest region is at depth, or we are interested just on the wavefield in a specific region to make localized updates to the model as in 4D seismic. There are different ways to solve this problem, such as redatuming the survey to a level above the target (da Costa et al., 2018) or building a local solver that calculates the wavefield in the target, changing the boundary conditions on the limits of target (Malcolm and Willemsen, 2016).

In condensed matter physics, the Patched Green’s Function (PGF) is a technique that allows the study electron transport in solids, such as nanoscale graphene waveguides (Power and Jauho, 2014; Settnes et al., 2015). The PGF technique is used to calculate the effect on the electron wave functions when holes are present in the graphene lattice. The way that PGF works is by applying a potential that removes the area of the hole from all the graphene structure. By making an analogy between the solid state lattice points and the discretized subsurface model, we implement a local solver that calculates the wavefield inside the target area and in receivers positions. We first calculate the Green’s function on the known region outside the target by removing it from the domain, and store these values. Then, we calculate the internal Green’s function by reconnecting the target area and filling into the entire domain. The latter step can be done at a much smaller cost than calculating the function for the entire model, allowing us faster updates.

Method and Theory

Impedance Matrix

The acoustic wave equation in frequency domain or Helmholtz equation for constant density is

\[ \nabla^2 \psi(x) + \frac{\omega^2}{c(x)^2} \psi(x) = F_0 \delta(x - x_s), \]  

where \( \omega \) is the wave frequency, \( c(x) \) is the wave speed at point \( x \) in space, \( \psi(x) \) is the wave field, and on the right-hand side is the source term that vanishes everywhere except at the point-like source located at \( x_s \), whose amplitude is directly related with the Fourier coefficient \( F_0 \).

A conventional method to solve equation (1) is approximating the laplacian using the finite-difference method. This leads to a linear system of equations where the variables are the amplitudes of wave field in a discretized domain. This linear system can expressed as

\[ \mathbf{G}^{-1} \psi = \frac{F_0}{h^2} \mathbf{U}^s, \]  

where \( \mathbf{G}^{-1} \) is called the acoustic impedance matrix (Marfurt, 1984). The vector \( \mathbf{U}^s \) is the source term being zero everywhere except on the source-containing cell \( s \), i.e.,

\[ U_i^s = \begin{cases} 1 & \text{for } i = s, \\ 0 & \text{otherwise}. \end{cases} \]  

By construction, the impedance matrix is tridiagonal in blocks, having 5 non-zero elements on each line if an 5-point stencil is used. The diagonal elements contain the information of the wave speed at each point of space whereas the off-diagonal ones are the connection terms between a point and its nearest neighbors that describe the wave propagation. The impedance matrix is the inverse of Green Function Matrix \( \mathbf{G} \), where each matrix element \( G_{ij} \) describes the wave propagation between sites \( j \) and \( i \).
Dyson equation

Dyson equation is a recursive equation that allows to calculate the Green Function $G$ of a system under a perturbation $V$ when is known the green function of the unperturbed system $g$. The Dyson equation can be stated as (Economou, 2006; Doniach and Sondheimer, 1974; Sheng, 2006)

$$G^{-1} = g^{-1} - V \Rightarrow G = g + gV$$

for any invertible matrices $G$ and $g$. The key advantage of using this equation becomes apparent when a single matrix element of the inverse of a matrix is required. Instead of calculating the entire matrix inverse, Dyson equation is often called upon to obtain a specific matrix element of the inverse without the need to calculate the entire matrix inverse. For large matrices this offers a substantial gain since the computational complexity in evaluating the inverse of a matrix of order $N$ grows as $O(N^3)$.

To illustrate how this tool can be applied to the FWI methodology, let us define the following quantities: $G$ is the GF of the whole system, $g$ corresponds to the GF of a system made of two separate parts that are completely disconnected, and $V$ consists of the matrix elements that bind these two parts together. This is schematically shown in Fig.1(a) for the illustrative case of $5 \times 5$ sites. Note that the system is indeed divided into two parts, hereafter referred to as outer (squares) and target (circles) regions.

The connection between the outer and target regions is shown in the figure by dashed lines localized between the internal border sites $I$, in red, and the external border sites $E$, in blue. In the context of Fig.1(a), $G$ represents the wave propagation across a system whose sites are fully connected whereas $g$ represents the solution to a system whose dashed-line connections are removed. Finally, the operator $-V$ simply acts as the connecting agent linking the parts. This is equivalent to representing the system as if the target region has been patched onto the outer domain. A less illustrative but more mathematical definition can be found in Moura et al. (2020)

Our interest is only on the wave field inside the target and at the receivers, so we need the terms $G_{ts}$ and $G_{rs}$ for each $t \in T$ and $r \in R$, and they can be obtained using the Equation 4,where results in the following equations.

$$G_{es} = (\mathbb{1} - g_{ce} V_{ei} g_{ie})^{-1} g_{es}, \quad (5)$$

$$G_{ts} = g_{te} V_{ie} G_{es}, \quad (6)$$

$$G_{rs} = g_{re} + g_{te} V_{ei} g_{ie} G_{es}. \quad (7)$$

Where we use the notation $A_{mn}$ to indicate a sub-matrix with appropriate dimensions formed by only the elements of the main matrices $A$ that refers to the set of sites indicated by the subscripts, and noting $V_{ei} = V_{ie}$.

To summarize the key ideas behind the PGF methodology, while the inverse of a sizeable impedance matrix is required to describe the wave propagation in a large system, the numerical complexity of this operation can be significantly reduced when that matrix is broken into two smaller sub-matrices. In particular, by writing the GF $G$ in terms of patched matrices $g$, as shown in Eqs.(5)-(7), the same wave signal normally obtained through standard numerical techniques can be found at a fraction of the time, as demonstrated next.

**Results**

We applied the proposed technique for a synthetic 2D model based on a real area of interest in Brazilian offshore. This velocity model is composed, from top to bottom, of a layer of water, followed by sediments, a thick layer of salt, the oil reservoir, and bed rock. The model has physical dimensions of 18 km $\times$ 7.5 km, divided in cells of side $h = 30$ m, resulting in $600 \times 250$ points. The model is showed in Fig.1(b). The small rectangle on the right-hand side highlights the $120 \times 45$ points area ($3.6$ km $\times$ 1.35 km) selected as the target. This area corresponds to 3.6% of the model.
Table 1: Computational time in seconds to perform 1 and 100 propagations in the target area.

<table>
<thead>
<tr>
<th>Method</th>
<th>1 propagation (s)</th>
<th>100 propagation (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conventional method</td>
<td>75</td>
<td>7500</td>
</tr>
<tr>
<td>PGF method</td>
<td>936</td>
<td>1253</td>
</tr>
</tbody>
</table>

Fig.2(a) shows the calculated GF for a wave of frequency 5 Hz, from a source located at cell (300, 2) to all the points in this velocity model directly solving the linear system of equation (2). For this model we applied the split out process in order to separate the target area from the model. After that we modified the velocity at target area subtracting 500 m/s. Figs. 3a) and 3b) show part of GF only at the target area, calculated in the conventional way and by the PGF method, respectively. In Fig. 2(b) we also plot receiver information along a line using the conventional method and PGF technique, showing the perfect numerical agreement between both methods.

To demonstrate the expected gain in processing time by using this technique instead of solving for the whole system every time, we calculate the spent time to calculate 1 and 100 propagations. These times are summarized in Table 1. The computational cost of the conventional method is the sum of spent time on LU factorization of impedance matrix plus the multiplication by the right hand size term times the number of propagations we need to perform. In PGF method, the computational cost is calculated by the time used to perform the split out operation, that is done once, plus the multiplication of time of fill in operation by the number of wave propagations performed. As shown in Table 1, for one propagation is not useful to use the PGF method because the split out method is more expensive than a propagation by conventional method. However, for problems where it is necessary to perform several propagations, as in case of FWI, we notice that the computational cost of PGF method is much lower than the conventional method. The computational gain showed in Table 1 is for the target showed in Fig. 1(b). If the size of target is increased the time of PGF method increases (Moura et al., 2020).

**Figure 1**
- a) Scheme representing the "split out" and "fill in" procedures.
- b) Velocity model used for numerical experiments showing the target of interest. The units in the figures are the positions of the cells on the matrix.

**Figure 2**
- a) Real Part of Green Function to all the points in this velocity model.
- b) Receiver information along a line using the conventional method and PGF technique. The units in the figures are the positions of the cells on the matrix.
Figure 3 Real Part of Green Function at target area, calculated by a) the conventional method and b) PGF technique. The units in the figures are the positions of the cells on the matrix.

Conclusions

In summary, we have demonstrated that by breaking a wave-propagating system into two separately disconnected parts, the computational complexity involved in the underlying calculations can be significantly reduced. By treating one of the parts as a target with changeable model parameters, it is possible to describe the wave motion in a complex system by inverting matrices that otherwise would be far too large to handle. This approach was implemented in target-oriented simulations of acoustic wave propagation with the focus to perform FWI in seismic readings.

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References


