Introduction

The dynamic ray tracing (DRT) is aimed at computing paraxial rays and the ray Jacobian along a given stationary (central) ray, yielding the relative geometric spreading and the identification and classification of possible caustics. The dynamic Eigenray is a natural continuation of the kinematic Eigenray performed with a similar, variational, finite-element approach. The DRT considers paraxial rays with infinitesimal shifts (deviations) from the central ray, and thus, the dynamic problem is linear (linearized). In this study, we follow the main ideas and concepts published more than one hundred years ago by Bliss (1916), on the Jacobi condition for variation calculus of parametric functionals. We adjust the general theory presented in this paper in order to derive the second-order DRT equation, and name it the Jacobi DRT equation. The formulation is based on the analysis of the second traveltime variation of a general paraxial ray in smooth, heterogeneous, general anisotropic elastic media. This leads to a linear, second-order, vector-form Jacobi equation. Hence, the Jacobi equation for dynamics is similar to the Euler-Lagrange equation for kinematics, but considers the second traveltime variation rather than the first variation. After the finite-element discretization, the weak formulation, and the use of the Galerkin method, the Jacobi equation set yields the same resolving matrix as the traveltime Hessian, obtained in the kinematic stage. The relative geometric spreading defines the magnitude of the Green’s function, while the accumulated number of caustics, with their order and concavity/convexity of the slowness surface (KMAH index) define the phase of the Green function: A line caustic yields a phase shift of $\pi / 2$ and a point caustic $\pi$. A slowness surface of a shear wave in anisotropic media may be convex or concave. In the case of caustics on a concave slowness surface, its contribution to the total KMAH index is negative (Klimes, 2010, 2014).

Type of Stationary Solutions

To analyze whether the path indeed delivers a minimum traveltime solution, we compute the eigenvalues of the global traveltime Hessian matrix along the stationary ray. This is the case where all the eigenvalues are positive. A (hypothetic) case where all the eigenvalues are negative indicates a maximum, which is unlikely for systems with multiple degrees of freedom (DoF). If one or more eigenvalues are negative, and the others are positive, it is a saddle point solution, which is an indication of the existence of caustics: zeros of the (signed) cross-section area of the ray tube (ray Jacobian) along the ray. However, this type of analysis cannot provide the number of caustics, their locations and their type: point or line, which are important for implementing the required phase correction of the Green function. This type of information requires actual solution of the DRT.

Jacobi ODE for Dynamic Ray Tracing

Given an initial guessed trajectory between two fixed endpoints $S$ and $R$, the condition for the trajectory to be a stationary traveltime path can be symbolically written as,

$$t = \int_{S}^{R} L(s) ds \to \text{stationary}, \quad L(s) = L\left[ x(s), r(s) \right] = \frac{\sqrt{\mathbf{r} \cdot \mathbf{r}}}{v_{\text{ray}}(\mathbf{x}, \mathbf{r})},$$

(1)

where $L(s)$ is the arclength-dependent Lagrangian, $\mathbf{x}$ is the ray position, and $\mathbf{x} = d\mathbf{x}/ds = \mathbf{r}$ is the ray velocity direction. The first and second variations of the traveltime read (Bliss, 1916),

$$\delta t = \int_{S}^{R} M_{1}(s) ds = \int_{S}^{R} \left( L_{x} \cdot \delta \mathbf{x} + L_{\mathbf{r}} \cdot \delta \mathbf{r} \right) ds,$$

$$\delta^{2} t = \int_{S}^{R} M_{2}(s) ds = \frac{1}{2} \int_{S}^{R} \left( \delta \mathbf{x} \cdot L_{xx} \cdot \delta \mathbf{x} + 2 \delta \mathbf{x} \cdot L_{x} \cdot \delta \mathbf{r} + \delta \mathbf{r} \cdot L_{\mathbf{r}} \cdot \delta \mathbf{r} \right) ds.$$

(2)

The vectors $L_{x}$ and $L_{\mathbf{r}}$ are respectively the spatial and directional gradients of the Lagrangian, and the matrices $L_{xx}$, $L_{\mathbf{r}\mathbf{r}}$ and $L_{x\mathbf{r}}$ are respectively the spatial, directional and mixed Hessians of the Lagrangian. Applying the Euler-Lagrange equation to the integrands of the first and second traveltime variations, we obtain the governing equations for the KRT and DRT, respectively.
\[ \frac{dL_T}{ds} = L_x \cdot \mathbf{L}_T = \mathbf{p} \]  \quad \text{Kinematic equation set (KRT)}
\[ \frac{d}{ds}( \mathbf{L}_{Tx} \mathbf{u} + \mathbf{L}_{Tu} \mathbf{\dot{u}}) = \mathbf{L}_{XX} \mathbf{u} + \mathbf{L}_{Xu} \mathbf{\dot{u}} \]  \quad \text{Dynamic equation (Jacobi DRT)}

The KRT delivers the stationary path, and the Jacoby DRT provides the normal shift vectors \( \mathbf{u}(s) \) of the paraxial ray, \( \mathbf{u} \cdot \mathbf{r} = 0 \), at any point of the stationary (central) ray. A general paraxial ray can represent a point-source ray, a plane-wave ray, or a combination of both. Two basic solutions are related to a point-source ray, and two others, to a plane-wave ray. The ray coordinates (RC), \( \gamma_i, i = 1, ..., 4 \), are introduced in order to distinguish between the paraxial rays,

\[ \mathbf{y}(\gamma_1, \gamma_2, \gamma_3, \gamma_4, s) = \mathbf{x}(s) + \gamma_1 \mathbf{u}_1(s) + \gamma_2 \mathbf{u}_2(s) + \gamma_3 \mathbf{u}_3(s) + \gamma_4 \mathbf{u}_4(s) \]  \quad \text{(4)}

The flow parameter (e.g., the arclength \( s \)) is also considered one of the RC.

**Initial Conditions for a Point-Source Ray**

Two normal vector shift solutions, \( \mathbf{u}_1(s) \) and \( \mathbf{u}_2(s) \), corresponding to two point-source initial conditions \( \mathbf{u}_{1,S} \) and \( \mathbf{u}_{2,S} \), are required to compute the ray Jacobian \( J(s) \). Matrix \( \mathbf{L}_{Tr} \) plays an important role in the DRT solution. It is a positive semi-definite matrix, with a single zero eigenvalue corresponding to the eigenvector \( \mathbf{r} \), and two positive eigenvalues. At the start point (the source \( S \)) the normal vector shifts \( \mathbf{u}_{1,S} \) and \( \mathbf{u}_{2,S} \) vanish, while the derivatives of these vector shifts (with respect to the arclength of the central ray) represent the normalized eigenvectors, corresponding to the two nonzero eigenvalues \( \lambda_{1,S} \) and \( \lambda_{2,S} \) of matrix \( \mathbf{L}_{Tr} \) at the start point \( S \). These vectors, \( \mathbf{u}_{1,S} \) and \( \mathbf{u}_{2,S} \), are normal to the ray and to each other,

\[ \mathbf{L}_{Tr,S} \mathbf{u}_{1,S} = \lambda_{1,S} \mathbf{u}_{1,S} \quad \mathbf{L}_{Tr,S} \mathbf{u}_{2,S} = \lambda_{2,S} \mathbf{u}_{2,S} \quad \mathbf{u}_{1,S} \times \mathbf{u}_{2,S} \cdot \mathbf{r}_S = 1 \quad \mathbf{u}_{1,S} = \mathbf{u}_{2,S} = 0 \]  \quad \text{(5)}

**Ray Jacobian and Relative Geometric Spreading**

The ray Jacobian is a mixed product of the two normal shift solutions and the ray velocity direction,

\[ J(s) = \frac{\partial \mathbf{y}}{\partial \gamma_1} \times \frac{\partial \mathbf{y}}{\partial \gamma_2} = \mathbf{u}_1(s) \times \mathbf{u}_2(s) \cdot \mathbf{r}(s) \]  \quad \text{(6)}

The ray Jacobian vanishes at the caustics. For a point caustic, both \( \mathbf{u}_1 \) and \( \mathbf{u}_2 \) vanish. For a line caustic, either one of the solutions vanishes, or the two vectors, \( \mathbf{u}_1 \) and \( \mathbf{u}_2 \), become collinear. The direction of the caustic line is, \( \mathbf{r}_c = \mathbf{w}_o \times \mathbf{r} \), \( \mathbf{w}_o \cdot \mathbf{r} = 0 \), where \( \mathbf{w}_o \) is the eigenvector corresponding to the zero eigenvalue of the transposed matrix, \( \mathbf{Q}^T = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{r}]^T \). The relative geometric spreading for the chosen RC is defined by,

\[ L_{GS}(s) = \frac{v_{ray,S} v_{ray}(s) \| J(s) \|}{\sqrt{v_{phs,S} v_{phs}(s) \lambda_{1,S} \lambda_{2,S}}} \]  \quad \text{(7)}

where the uppercase \( S \) means the source, and the lowercase \( s \), an arbitrary point along the central ray. The relative geometric spreading is an objective physical characteristic, independent of the choice of the RC, and reciprocal (insensitive to the source and receiver swap). This is not so for the ray Jacobian. Therefore, equation 6 is valid only for the specific choice of the RC listed in equation 5. For isotropic media, equation 7 simplifies to, \( L_{GS}(s) = v_S \sqrt{|J|} \), where \( v_S \) is the velocity at the source.

**Jacobi DRT Finite-Element Solver**

We apply locally the weak formulation and the Galerkin method to the Jacobi ODE set,

\[ \int_{s_{fin}}^{s_{ini}} \frac{d}{ds} \left( \mathbf{L}_{Tx} \cdot \mathbf{u} + \mathbf{L}_{Tu} \cdot \mathbf{\dot{u}} \right) w(s) ds = \int_{s_{ini}}^{s_{fin}} \left( \mathbf{L}_{XX} \cdot \mathbf{u} + \mathbf{L}_{Xu} \cdot \mathbf{\dot{u}} \right) w(s) ds \]  \quad \text{(8)}
where \( s_{\text{ini}} \) and \( s_{\text{fin}} \) are the values of the arclength at the nodes of the finite element, and \( w(s) \) is the weight function. We change the arclength \( s \) to the internal parameter of the finite element, \(-1 \leq \xi \leq +1\), and we apply integration by parts in equation 8. This removes the second derivative of the normal shift \( \ddot{u} \),

\[
\begin{align*}
\int_{\xi=-1}^{\xi=1} \left( L_{\text{tx}} \cdot u + L_{\text{tr}} \cdot \ddot{u} \right) \frac{dw}{d\xi} d\xi + \\
\int_{\xi=-1}^{\xi=+1} \left( L_{xx} \cdot u + L_{xr} \cdot \ddot{u} \right) w \frac{ds}{d\xi} d\xi
\end{align*}
\]

(9)

Introducing the Hermite interpolation polynomials to represent the normal shift and its derivative through their nodal values converts the differential equation into algebraic. It turns out that the global resolving “stiffness” matrix for DRT coincides with the global traveltime Hessian of the stationary ray, computed on the last iteration of the KRT. Hence, performing combined kinematic and dynamic analyses becomes effective.

**Numerical Example**

We present a numerical example for an anisotropic model, where both kinematic and dynamic characteristics of stationary ray paths have been computed. In this example, eight three-node finite elements were used to present the ray path. Obviously, for real field examples, the number of nodes may be much higher.

Figure 1 shows (a) the ray path, (b) the relative geometric spreading, and (c) a normalized geometric spreading for an ellipsoidal orthorhombic model with a constant tilted reference velocity gradient. An ellipsoidal medium is obtained from a general orthorhombic medium by setting the three shear stiffness components and the three intrinsic anellipticities (Alkhalifah, 2003) to zero: \( C_{44} = C_{55} = C_{66} = 0 \), and \( \eta_1 = \eta_2 = \eta_3 = 0 \), so that only three medium parameters remain. The Christoffel equation of such a model is given by,

\[
A_1^2 p_1^2 + B_1^2 p_2^2 + C_1^2 p_3^2 = 1 ,
\]

(10)

where \( A_1(x), B_1(x), C_1(x) \) are spatially varying axial velocities. The model belongs to the FAI class (Factorized Anisotropic Inhomogeneous) (Červený, 2001), such that the axial velocities are proportional to the reference (background) velocity \( v(x) \).
\[ A_v(x) = \lambda_a v(x), \quad B_v(x) = \lambda_b v(x), \quad C_v(x) = \lambda_c v(x), \quad v = v_a + k \cdot x. \] (11)

The coefficients \( \lambda \) are related to Tsvankin (1997) parameters \( \varepsilon_1 \) and \( \varepsilon_2 \)

\[ v_P(x) = \lambda_c v(x), \quad \sqrt{1 + 2\varepsilon_2} = \lambda_a, \quad \sqrt{1 + 2\varepsilon_1} = \lambda_b \] (12)

where \( v_P \) is the varying vertical (compressional) velocity. We set the values, \( \lambda_a = 1.25, \lambda_b = 1.15, \lambda_c = 1 \). The reference velocity has a constant tilted gradient in \( x_1x_3 \) plane, \( k = [0.2 \ 0 \ 0.8] \text{s}^{-1} \), and the origin of the reference frame is located at the midpoint, where the velocity is \( v_a = 3 \text{ km/s} \). The offset is \( h = 10 \text{ km} \). The analytical solution exists for this simple type of anisotropy: the ray path is elliptic: the semi-axes and the location of the center can be computed. We obtained highly accurate agreement between the analytical and numerical ray trajectory shown in Figure 1a, the traveltime, arclength of the path and parameter \( \sigma \). Different colors show segments of the path that belong to eight finite elements. Figure 1b shows the relative geometric spreading \( L_{GS}(s) \) vs. the arclength of the central ray. The solid line in this plot corresponds to the source located to the left of the midpoint, and the receiver to the right. The dashed line shows the relative geometric spreading for the source and receiver swapped. Note that the solid and dashed lines accept different values along the path, but they converge at the final point of the arclength. This is due to the reciprocal nature of the relative geometric spreading. Figure 1c shows the normalized relative geometric spreading (also for the direct and reverse locations of the source and receiver), defined as \( L_{GS} / \sigma \), where parameter \( \sigma \) is defined by \( d\sigma = v_{ray} ds = v_{ray}^2 d\tau \). For isotropic media with a constant velocity gradient (not necessarily vertical), \( L_{GS} = \sigma \), and the normalized relative geometric spreading is 1. As we see in Figure 1c, this is not so for FAI anisotropy with a constant gradient of the reference velocity.

Conclusions

We derive an original dynamic ray tracing (DRT) method in order to compute relative geometric spreading along a stationary ray and to identify and classify possible caustics. The theory and implementation are valid for 3D smooth heterogeneous general anisotropic media and for all types of wave modes. The proposed DRT equation set governs the second traveltime variation for stationary ray paths, formulated as the Jacobi, second-order, linear ODE set. By using the weak formulation, with the Galerkin method, the ODE set is locally reduced to the first-order, linear algebraic, weighted residual equation set. Instead of the traditional initial-value numerical integration approach (e.g., Runge-Kutta), where the derivatives are approximated by finite differences, we apply a variational approach and solve the ODE set with an accurate finite-element implementation using the same Hermite interpolation scheme used for the kinematic Eigenray method. One of the main advantages of the proposed method is that the resolving matrix of the Jacobi DRT linear algebraic equation set coincides with the traveltime Hessian matrix along the stationary ray. This traveltime Hessian matrix has already been evaluated as part of the kinematic solution and hence should not be recomputed, making the implementation of the proposed method straightforward. We have successfully demonstrated the high accuracy of the proposed method in a number of isotropic and anisotropic benchmark problems.

References