Introduction

An important part of the Full Waveform Inversion (FWI) process for seismic imaging is the direct solution of wave equations on domains with no natural boundaries. For the numerical solution of such equations we have to impose some kind of boundary conditions which, however, should not cause artificial reflections. There are several methods which have been proposed to minimize these reflections in the literature. In this work we have implemented and compared different methods for handling the boundaries when solving acoustic wave equations with finite difference schemes. The methods can be divided in two types: absorbing boundary conditions, such as Clayton and Engquist scheme or Higdon methods, and absorbing boundary layers, including sponge layers, hybrid absorbing boundary conditions (HABC) and perfectly matched layer (PML) schemes. We have compared the methods on a close to realistic test case, with a synthetic velocity model typical for a pre-salt section at Santos basin in Brazil. Besides the popular PML approach, Higdon and HABC have shown to be interesting alternatives, depending on the precision required and computational costs.

The methods employed

a) The acoustic wave equation: We consider here the acoustic wave equation on a rectangular domain \( \Omega = [x_I, x_F] \times [z_I, z_F] \). Let \( \partial \Omega \) denote the boundary of \( \Omega \), as represented in Figure (1).

\[
\begin{array}{ccc}
\partial \Omega_1 & \Omega & \partial \Omega_3 \\
\partial \Omega_2 \\
\end{array}
\]

Figure 1 Domain and boundaries representation.

The equation for the unknown \( u(x, z, t) \) is given by:

\[
\frac{\partial^2 u(x, z, t)}{\partial t^2} - c^2(x, z) \Delta u(x, z, t) = f(x, z, t),
\]

where \( \Delta \) denotes the spatial Laplacian operator. The function \( f \) accounts for the forcing terms and \( c(x, z) \) for the local wave speed. The initial values \( u(x, z, 0) \) and \( \partial u(x, z, 0)/\partial t \) for \( (x, z) \in \Omega \) are also given.

We employ a standard second order finite difference discretization of this equation in our numerical experiments.

b) Absorbing boundary conditions: These are boundary conditions added directly to the equations, which ideally would not cause reflections of outgoing waves. The Clayton first (A1) and second (A2) order conditions (at \( x = x_F \), corresponding to \( \partial \Omega_3 \) in Figure 1) are given by (cf. (Clayton, 1977))

\[
A1: \quad \frac{\partial u}{\partial x} + \frac{1}{c} \frac{\partial u}{\partial t} = 0 \quad \text{and} \quad A2: \quad \frac{\partial^2 u}{\partial x \partial t} + \frac{1}{c} \frac{\partial^2 u}{\partial t^2} - \frac{c}{2} \frac{\partial^2 u}{\partial z^2} = 0,
\]

being analogous at the other boundary lines.

At the same boundary \( \partial \Omega_3 \) the Higdon conditions of order \( p \) are given by (cf. (Higdon, 1987))

\[
\left[ \prod_{k=1}^{p} \left( \cos \alpha_k \frac{\partial}{\partial t} - c \frac{\partial}{\partial x} \right) \right] u = 0,
\]

where the incidence angles are such that \( |\alpha_k| < \pi/2 \). The first order Higdon condition with \( \alpha_1 = 0 \) corresponds to Clayton’s A1 condition, while condition A2 should be equivalent to Higdon second order
condition with both incidence angles equal to 0, although this depends on the discretization employed. In this sense, Higdon conditions generalize the Clayton conditions.

c) Absorbing boundary layers: In order to minimize wave reflection this kind of methods incorporate an extra region surrounding the domain, where the outgoing waves should be sufficiently damped in order not to interfere with the solution at the interior of the domain. One of the simplest methods to do this was proposed in (Cerjan, 1985). In this case, the solution itself should be multiplied by an exponentially decaying function in the extra boundary layer, varying from 1 at the domain boundary to almost zero in the end of the extra layer. The computations are carried out on an extended domain of the form \( \tilde{\Omega} = [x_I - Lx_I, x_F + Lx_F] \times [z_I - Lz_I, z_F + Lz_F] \) and the multiplying function at the layer extending the boundary \( \partial\tilde{\Omega}_I \) is of the form \( G(x) = e^{-\lambda(x-x_F)^2} \). Another simple scheme, employing the same kind of extended domain \( \tilde{\Omega} \) was proposed in (Sochaki, 1987). In this method equation (1) is modified to

\[
\frac{\partial^2 u(x,z,t)}{\partial t^2} + 2a(x,z)\frac{\partial u(x,z,t)}{\partial t} - c(x,z)^2 \Delta u(x,z,t) = f(x,z,t),
\]

where the function \( a(x,z) \) vanishes inside \( \Omega \) and increases along the boundary layer. Several forms have been proposed in (Higdon, 1987), with linear, cubic or exponential growth for instance.

The probably most popular methodology with absorbing layers was initially proposed in (Berenger, 1994) and further developed and used in many applications. The technique is derived by analytic continuation of the waves, such that they will exponentially decay when leaving the domain. We follow here the formulation proposed in (Grote, 2010). Equation (1) is replaced by the system:

\[
\frac{\partial^2 u(x,z,t)}{\partial t^2} + (\xi_1(x,z) + \xi_2(x,z))\frac{\partial u(x,z,t)}{\partial t} + \xi_1(x,z)\xi_2(x,z)u(x,z,t) = c(x,z)^2 \Delta u(x,z,t) + \frac{\partial \phi_1(x,z,t)}{\partial x} + \frac{\partial \phi_2(x,z,t)}{\partial z} + f(x,z,t)
\]

\[
\frac{\partial \phi_1(x,z,t)}{\partial t} = -\xi_1(x,z)\phi_1(x,z,t) + c(x,z)^2(\xi_2(x,z) - \xi_1(x,z))\frac{\partial u(x,z,t)}{\partial x}
\]

\[
\frac{\partial \phi_2(x,z,t)}{\partial t} = -\xi_2(x,z)\phi_2(x,z,t) + c(x,z)^2(\xi_1(x,z) - \xi_2(x,z))\frac{\partial u(x,z,t)}{\partial y}
\]

where \( \phi_1 \) and \( \phi_2 \) are auxiliary functions, with zero initial values. The damping functions \( \xi_1 \) and \( \xi_2 \) vanish inside the domain \( \Omega \). Therefore, the equation is actually only modified in the boundary layer. The damping functions adopted here have a logistic shape, growing from zero in the boundary of \( \Omega \) to a value \( \xi = 40 \) at the end of the boundary layer, with an inflection point in the middle of the layer (see (Grote, 2010) for the expression). The last technique we investigate consists of the hybrid absorbing boundary conditions, as proposed in (Liu, 2018). The field at a new time-step of the model within the boundary layer is the combination of two fields obtained from the previous time-step. One is computed in the whole domain \( \tilde{\Omega} \) - which includes the boundary layer - with the discretized form of equation (1), with homogeneous Dirichlet boundary condition at the most external boundary. The other field differs in the boundary zone, where at each grid line the solution is computed as if the domain ended there, employing an absorbing boundary condition (Clayton’s A2 or Higdon, by instance). A linear convex combination of the two fields along the boundary layer, with the weights given to the latter varying from 0 to 1, builds the solution at the next time-step. Actually, as pointed out in (Liu, 2018), a quadratic variation of the weights (from 0 to 1) produces better results than the linear one, being therefore adopted here.

**Numerical Experiments with a synthetic velocity model of a pre-salt section at Santos basin**

For a more realistic test of the several boundary conditions we employ a synthetic velocity model, representing a typical pre-salt section in Brazil. The domain is set according to the following parameters: \( x_I = z_I = 0km, x_F = 70,3km \) and \( z_F = 9,920km \), where \( x \) denotes the length and \( z \) the depth of the geological region considered, which essentially consists of 4 vertical layers (cf. Figure (2)).
The minimum speed is approximately 1.5 km/s on the water layer above and the maximum value is around 7 km/s on the bottom layer. For our simulations we start from zero initial conditions, triggered by a Ricker source term with peak frequency $f_0$ of 5 Hz, located in the middle of the domain in the $x$ direction and close to the surface. On the top boundary (surface) ($\partial \Omega_4$), as in Figure 1) we use homogeneous Neumann boundary conditions, while the absorbing boundary conditions will be used at the lateral and bottom boundaries. We locate the ‘receivers’ at $y = 0.032Km$ of depth, monitoring the value of the displacement at these positions along the simulation. From these values we emulate a seismogram.

For comparing the several schemes, we first defined a reference solution, computed on the full domain, extending the bottom layer (with the same constant velocity) up to $z_F = 19.840Km$. In this way, any possible reflection from the bottom, will be at least postponed. This reference solution is computed with the PML scheme, with the last 30 grid points in each direction defining the damping layer. The model is run for 20s with a time-step of approximately $1.8 \times 10^{-3}$ s, corresponding to a CFL around 0.4.

For testing the schemes we used a smaller domain with 40 km length (from $x_l = 15,150km$ to $x_F = 55,150km$) and in depth from $z_l = 0$ to $z_F = 8,928km$. The boundary layers for the PML, HABC and Sochaki Damping model used 30 extra grid points in each direction, except on the surface. Since the reference run extends for more 15 km on each side of the domain, we will not have boundary effects on this run in the region used for the comparisons.

We first show in Figure (3) a snapshot of the solution (displacement) at 3.6 s. At this instant, the wave solution has already reached the bottom, and we can observe reflection effects, especially with Clayton’s A2 and Sochaki Damping. In the figure we display the relative errors, between the solutions with the several schemes and the reference run. In Figure (4) we show the evolution of the maximum errors in the displacement along the integration. We can observe that the best schemes are PML, HABC and Higdon.

Finally, in Figure (5) we show the computed seismogram and display the relative maximum differences in the seismograms computed with the various methods, where again we can observe which ones perform better.

**Figure 2** Synthetic Velocity Profile.

**Figure 3** Displacement at $T = 3.6$ and relative errors with the various schemes.

**Conclusions**

Our numerical results on a close to real data test show that simple boundary conditions as Sochakis damping scheme, or even Clayton’s BC are not adequate to avoid artifical wave reflections. The PML
scheme produces the best results, but the Hybrid and Higdon schemes deliver good results at competitive computational costs, and may be preferable when a somewhat lower accuracy is acceptable.

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References


