Introduction

A number of geophysical phenomena are captured in partial differential equations (especially in the second order). A usual approach for solving a simple differential equation is by integration where an exact/analytical solution to the differential equation can be computed. But even in the case of slightly complicated differential equations, numerical integrations have to be adopted to find an approximated solution. There are analytical solutions for some geophysical problems, especially for homogeneous models in 1D, 2D or 3D. However, the reality of geophysical problems is the existence of complex geometry and geology whose analytical solutions do not exist and solutions by the method of numerical integration are not trivial. Therefore, the numerical approximation of the differential equations is used in the form of finite difference and finite elements approximations.

The finite difference, for example, is an approximation from the Taylor’s series expansion whose accuracy depends on the order (number of terms used and the ones that are truncated). Thus the finite difference scheme is inherently burdened with a truncation error. Essentially, the finite difference coefficients can be thought as a differentiation matrices operator although it is an approximation (e.g. Trefethen, 2000). Chebyshev, pseudo-spectral and full spectral methods are among the other methods in which differentiation matrices operators have been generated. The three approaches use some form of transformation: cosine for the Chebyshev and the Fourier transform for the spectral domain. In the Fourier domain the spatial and temporal derivatives become complex wavenumber and angular frequency respectively. The spectral differentiation matrices operator are analytical (accurate) because they are infinite order finite difference discretization (Igel, 2017). The term pseudo-spectral method suggests that only a single domain (either space or time) is transformed into the spectral domain and the other domain is discretized by the finite difference method (Igel, 2017). In other words, the total differentiation matrices operator for the pseudo-spectral method is not completely accurate since the finite difference introduces an approximation into it. But the full spectral domain performs Fourier transform on both the temporal and spatial components of the field thus offering a complete accurate differentiation matrices operator. However, the Fourier transforms assumes that the function to be sampled is periodic, and discretized at equal grid size, and therefore careful sampling theorem must be applied in order to prevent an aliasing effect.

The finite difference methods have been used to solve the high frequency GPR Helmholtz equations (Irving and Knight, 2006). Trefethen (2000) solved variable coefficient wave equation by the pseudo-spectral method. Yoon (1997) applied the full spectral method to solve the acoustic scalar wave equation. Following the methodology in Yoon (1997) we present the modeling of the GPR full waveform by the spectral method. Both the temporal and spatial coordinates are transformed to the complex Fourier domain; the convolution of the spectral (complex) differentiation matrices operator with the medium parameter will be performed; and the set of linear equations will be inverted to produce the transverse electric (TE) and the transverse magnetic (TM) modes of the GPR full waveform. Inverse Fourier transformed will be carried out only on the wavenumber back to space in order to study the propagation of the GPR full waveform in the space-frequency domain.

Theory and Method

Faraday’s and Ampere’s laws of electromagnetic (EM) in time \((t)\) and space \((r)\) domains are given in equations 1 and 2 respectively (Ogunbo, 2011):

\[
\nabla \times \mathbf{E}(r, t) + \mu \partial_t \mathbf{H}(r, t) = -\mathbf{J}^m(r, t),
\]

\[
-\nabla \times \mathbf{H}(r, t) + (\sigma(r) + \varepsilon(r))\partial_t \mathbf{E}(r, t) = -\mathbf{J}^e(r, t),
\]

where \(\mathbf{E}\) is the electric field in \(V/m\); \(\mathbf{H}\) is the magnetic field in \(A/m\); \(\sigma\) is the conductivity \((S/m)\); \(\varepsilon\) and \(\mu\) are respectively the permittivity \((F/m)\) and permeability \((H/m)\) of the medium; electric and magnetic sources are given as \(\mathbf{J}^e\) and \(\mathbf{J}^m\) respectively, and \(r\) represents the \((x, y, z)\) spatial coordinate.

For a 2D geometry the medium parameters do not change along the y-axis. Equations (1) and (2) can be decomposed into the transverse electric (TE) and transverse magnetic (TM) modes. With respect to
The y-axis, the TE-mode fields are $E_x, H_z$ and $H_y$ and the TM-mode fields are $H_x, E_y$ and $E_z$ (Simpson and Bahr, 2005). The scalar wave solution for the TM-mode $E_y$ from a source function $f(x, z, t)$ is:

$$\partial_t^2 E_y(x, z, t) + \mu_e(x, z) \partial_t^2 E_y(x, z, t) - \mu_e(x, z) \partial_t E_y(x, z, t) = f(x, z, t).$$  (3)

The PML absorbing boundary condition can be directly incorporated in equation (3) following the Berenger (1994) implementation: $\tilde{\partial}_r = \frac{r}{\eta_r} \partial_r$, where in the 2D geometry $r = x$ or $z$ and $\eta_r = 1 + \frac{i \nu}{\omega}$; $\nu_r$ is the stretched medium velocity. In the full spectral domain (frequency and wavenumber), $\partial_t = i \omega, \partial_r = i k_r$; where $i = \sqrt{-1}, \omega = 2 \pi f$, $f$ is the frequency (Hz) and $k_r$ is the wavenumber along the r coordinate. Putting all of these together, keeping only the high frequency displacement current and ignoring the conduction current (loss), and using the convolution theorem on the Fourier transform of the product of two signals the full spectral domain formulation for equation (3) is:

$$ (\tilde{V}(k_x, k_z) ** (ik_x)^2 + \tilde{V}(k_x, k_z) ** (ik_z)^2 - \tilde{\eta}(k_x, k_z) ** (i \omega)^2) \tilde{E}_y(k_x, k_z, \omega) = \tilde{F}(k_x, k_z, \omega),$$  (4)

where $**$ is the 2D convolution, $\tilde{V}(k_x, k_z)$ is the 2D Fourier transforms of $v^2(x, z) = 1/(\mu_e(x, z));\tilde{\eta}(k_x, k_z)$ is the 2D Fourier transforms of $\eta^2(x, z)$, and $\tilde{F}(k_x, k_z, \omega)$ is the 2D Fourier transform of $v^2(x, z) \eta^2(x, z) f(x, z, t)$.

Casting equation (4) in the matrix-vector form we have

$$\tilde{M} \tilde{E}_y = \tilde{F},$$  (5)

where $\tilde{M}$ is the complex impedance matrix and $\tilde{E}_y(k_x, k_z, \omega)$ can be computed by direct-matrix inversion. The 2D inverse Fourier transform of $\tilde{E}_y(k_x, k_z, \omega)$ will eventually give $E_y(x, z, \omega)$ which is the solution to equation (3) in space-frequency domain.

Assuming the source function in equation (3) is an impulsive response then its solution is the 2D Green’s function, $G_{2D}(x, z, \omega)$. Hence the exact electric field when the true source function is known is simply the Green’s function scaled by the source term. Therefore for $f = -i\omega \mu f_0$, the exact TM-mode $E_y$ equals (Lavoue, 2014):

$$E_y(x, z, \omega) = -i \omega \mu f_0^0 G_{2D}(x, z, \omega),$$  (6)

and the TE-mode $H_y$ field is:

$$H_y(x, z, \omega) = f_0^0 \partial_x G_{2D}(x, z, \omega) - f_0^0 \partial_z G_{2D}(x, z, \omega),$$  (7)

In the full spectral domain

$$H_y(i k_x, i k_z, \omega) = f_0^0 i k_x G_{2D}(i k_x, i k_z, \omega) - f_0^0 i k_z G_{2D}(i k_x, i k_z, \omega),$$  (8)

For a homogeneous medium the $G_{2D}(x, z, \omega) = 0.25 i H_0^{(1)}(k \sqrt{x^2 + z^2})$ where $H_0^{(1)}$ is the Hankel function of the first kind and order zero; $k = \omega/v$.

Example 1

The common modeling parameters for all of the sources presented here are: frequency of source excitation of 1 GHz, the horizontal and vertical spacing of $dx = dz = 0.03$ m, and the computational domain has 101 points along the x and z directions. We vary only the relative permittivity ($\varepsilon_r$) models. For a homogeneous medium of unit $\varepsilon_r = 1$ (air) with source at the center of the model, the real components of $E_y(x, z, \omega)$ and $H_y(x, z, \omega)$ results are shown in Figures 1a and b. In Figures 1c and d, a profile across the center is taken and finite difference – 5-point approximation in Ajo-Franklin (2005) – and the full spectral results are compared with the analytical solution. While there is an accurate match between the full spectral and analytical results, there is a visible departure of the finite difference result from the analytical solution. The use of higher order approximation, say 9-point (Jo et al., 1996), can improve the accuracy but it will still be an approximation compared Moreover, increasing the approximation order beyond 9-point presents a prohibitively unaffordable cost on the computer memory resources (Ajo-Franklin, 2005).
Figure 1 Real components of GPR full waveform propagation at frequency 1 GHz for $\varepsilon_r = 1$ (a) TM $E_y$ (b) TE $H_y$ (c) center profile for $E_y$ (d) center profile for $H_y$.

Example 2

A second example is a 2-layered model $\varepsilon_r = 2$ and 5 for the first and second layers respectively) with an air layer ($\varepsilon_r = 1$). Figure 2a describes the model with source position around 1 m. In Figure 2b, the real component of the TM $E_y$ wavefield propagation has been captured. The wavefield propagates outwardly from the source towards the permittivity boundary interfaces where reflected and transmitted energies are observed. The transmitted energy above the 0 m, are the events representing the direct airwave. The airwave is the fastest since the EM wave travels at the speed of light in the air. More so the speed of propagation is directly proportional to the inverse of the square root of the relative permittivity.
Figure 2 (a) 2-layered permittivity model with air layer, (b) $E_y$ propagation in (2a)

Conclusions

GPR full waveform modeling in 2D has been solved in the full spectral domain. It exploits the full accuracy provided by the spectral differentiation operator which completely avoids the truncation error fully and partially inherent in the finite difference and pseudo-spectral approaches. Modeling results show the superior accuracy of the full spectral modeling over the finite difference method.

Acknowledgements

We would like to thank the Korean Ministry of Education for Brain Korean (BK) scholarship for this work.

References


