The economic structure of a one-period security market

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Abstract

We present an analytical model of a one-period financial market for one asset. All traders learn by least squares regression, and interlocking expectations drive price towards return. We find that market shares of different trader types at economic equilibrium are proportional to the salience and scarcity of their information. Efficiency is a property generated by the market itself rather than a property inherited from a subset of fully informed traders. Market efficiency is not complete, and the residual discrepancy between price and return provides the return to the trading agents and resolves the Grossman-Stiglitz paradox.

Keywords: regression market, coefficient space, least squares learning, Grossman Stiglitz paradox, coefficient model

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1. Introduction: the microeconomic theory of financial markets

To the extent that price approaches value in most models of the market for a single security, it is because the models define a class of fundamental analysts who know with reasonable accuracy what the value of the security is, and force price in that direction.[2]

Here, by contrast, we analyse what we term a 'regression market'. Traders are treated symmetrically in that all traders are least squares learners, although their individual strategies may differ. There are no fundamental analysts who receive private signals about security value. Our reasoning is as follows. Suppose trader *j* receives an exogenous signal x_j

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[2] Considering a few examples in this area, in Grossman Stiglitz (1980) there are two classes of investor, 'informed' and 'uninformed'. Return is $u = \theta + \varepsilon$, where investors pay cost *c* to observe θ. Bray (1982) introduces least squares learning, but only for the uninformed investors to play catch-up. The informed investors are assumed to know the expected value of security return through rational expectations. Brock and Hommes (1997) construct a model where all the agents use a predictor function for price in the next period using an estimator, the accuracy of which depends on its cost. One class of agents uses a costly rational expectations operator which generates the future price accurately, and a second class uses costless adaptive expectations. Turning to more recent contributions, Vives (2014) supposes that trader *i* receives a signal $s_i = \theta_i + \varepsilon_i$ about value θ which is specific to the agent which is independent of the price, but correlated with the signals received by other agents. Ali (2018) constructs a model wherein agents receive payoff $u(a,w)$ in each period according to state of the world *w* and their action *a*. Each 'speculative' (i.e. informed) trader can conduct an experiment *X* which offers information about the state of the world *w* and consequent payoff $V(w)$ at cost $c(X,\theta)$ where θ is the cost of that kind of information for that particular agent. There is also a class of 'noise' (i.e. uninformed) traders who do not receive information. In their model, Lambert Ostrovsky and Panov (2018) allow each 'strategic' (i.e. informed) trader i to observe privately a multidimensional signal θ_i. At least some of the signals are correlated with final value *v*. It is notable that little consensus has emerged as to the information generating mechanism, the notation, or even the terminology for the various classes of investor.

about security return *y*, i.e. $y = \mathrm{f}\left(x_j\right)$. This signal alone is not consistent with profitable trading, because profit is determined not by gross return *y* but by net return $y - p$, where *p* is the price paid by the trader for the security. The trader needs to know to what extent the information which they receive is already embedded in the price *p*, in other words the trader needs to know function $p = \mathrm{g}\left(x_j\right)$. Function g depends on three things: the number of other traders who also have signal x_j, the degree to which signal x_j is correlated with other signals x_k which are also in use, and the financial resources of other agents relative to *j*. Information on these factors, and consequently function g, is not directly accessible to traders in any real-world context. It follows that trader *j*'s net return

$$y - p = \mathrm{f}\left(x_j\right) - \mathrm{g}\left(x_j\right) \tag{1}$$

cannot be known with precision *even if function f is known precisely*. And it is open to question whether function f can be known precisely. Real-world methods of analysis appear to be concerned with uncovering empirical regularities which can be exploited, not deterministic formulae. Even analysis derived purely from accounting data can only be assessed in the light of experience.

We conclude that the trader must necessarily use an estimation technique to discern the relationship between signal x_j and the target variable $y - p$. Here we suppose that investors use ordinary least squares (OLS) regression. Of course many other estimation techniques are available, but the general properties of any measure of central tendency are captured by OLS. We suppose that the simple regression assumption does not materially restrict the generality of the analysis. A regression market model has scientific advantages in that all trading agents are structurally the same and use a well-defined method of information processing, so there are no black-box processes lying outside the model.

We take it that agents are heterogeneous in that they use different data series in their estimations. We show that price converges to value in the regression market, so that efficiency is an intrinsic property of the market itself rather than a property inherited by the market from well-informed agents. Essentially the reason for this is that the return expectations of heterogeneous agents sum to zero, so price is forced in the direction of the estimates of the agents who have most recently estimated. The process does not rely on how much money is behind each new estimate, although the trajectory of convergence does depend on this weight of money.

As investors in our model are using OLS learning, the market can be characterized using linear algebra. This allows explicit algebraic expressions to be found for market characteristics. An important feature of a regression market is that the data set itself drops out of the analysis, and we are able to operate in an *N*-dimensional space of the price and value coefficients. Price can be understood as a vector rather than a single point, and as an *object* as that term is used in computing. A computing object can be defined as a reified entity with its own well-defined properties and methods for interacting with other entities. Such an understanding is as distinct from the view that price is an immediate consequence of supply and demand decisions.

A difference between the model developed here and an adaptive expectations or least squares learning *macroeconomic* model is that within such models the realised value of the parameter of interest can be directly observed by the agents. Such models are typically closed by an equation of the form:

$$Change\ in\ Inflation\ expectations = k\left(actual\ Inflation - expected\ Inflation\right) \quad (2)$$

By contrast, in a financial markets model, agents never observe the parameter which governs the return to the security and the mechanism of convergence is quite different. It relies on the fact that the security demand sums to zero, a circumstance which is not relevant to macroeconomic variables. The mechanism of convergence is laid out in the price change principle given below.

The balance of the paper is organized as follows. In section 2 we present the premises of the model. In section 3, we demonstrate that a regression market can make a price, and that the expected price converges to a value related to security return. To proceed further we need to analyze the model more deeply to determine the equilibrium price and the price regression coefficient accurately. This analysis is carried out in Section 4 and may be skipped on a first reading. With these preliminaries complete, in Section 5 we calculate the short and long term profit of each type of trader and present the economics of the security market. We show that each agent is rewarded by the market according to the relevance and scarcity of the information they provide, and that the behaviour assumed of agents is economically rational and sustainable. Finally in the conclusion we draw out the conceptual implications of the study.

2. Premises of the coefficient model

Premise 1: One period model. A one-period model is the simplest possible model of a security market. There is a security which trades at price p at the start of a period, makes an uncertain payment of y per unit at the end of the period, and retires. That is, the market is the same as a futures or options market, but without the margin calls. The return y and the price p can be negative, e.g. a contract to supply something. We do not consider the time value of money in the one-period model so that we can focus on the motivations and profit realisations of the traders without the complications of capital return.

Premise 2: Heterogeneous learning. We distinguish three types of trading strategy. The same agent can use several strategies but they are treated separately.

Natural business. In capital markets, investors and borrowers wish to deposit or borrow money at the risk premium rate. In derivative markets such as futures and swaps, businesses wish to hedge exposures incurred elsewhere. For these agent types, fluctuations in market price are a small consideration and they are prepared to accept the market price as sufficiently accurate for their purposes. Such agents are often referred to as noise traders within the scientific literature, following Black (1986), because their trades are uninformed by specific information about the security, and do not add to the information incorporated into the price. However it is misleading to see these traders as irrational because they participate for reasons other than trading profit. We account for these non-trading motives by assuming that each natural business agent has a limit on the average trading loss which they are prepared to pay to participate in the market.

Traders do not have an investing or hedging motive for trading in the market. They take short-term positions which they later reverse, in order to exploit inaccuracies in the price. As price is endogenous we distinguish trading strategies using price from strategies using non-price data.

Price strategy: Each investor is aware that other investors are using different but useful information. We suppose that most traders attempt signal extraction from the price by carrying out a price regression. This is a simple OLS regression on security net return $y - p$ using price as the explanatory variable. Agents do not include other variables with price in the same regression, because price does not have a consistent relationship to other variables and does not generate stable estimates if combined with other variables.

Non-price data strategy: This is a multivariate OLS regression on security net return $y - p$ using exogenous data series as the explanatory variables. In general different investors use different explanatory variables, although as a special case all data traders can use the same data. Agents may include a constant term in data regressions. In this model, the constant is treated as an iota vector $\boldsymbol{\iota}$ and not given special treatment.

Remark: The assumption that data watchers use some but not necessarily all variables is consistent with the Clements and Hendry (2005) finding that short models (i.e. those which omit some explanatory variables) often outpredict longer ones, notwithstanding the biased estimation which a short model entails, particularly where the model specifications differ from the underlying data generation mechanism.

Remark: The least squares learning assumption may seem to be unduly restrictive, but virtually any method used in practice to estimate stock market returns can be rendered at least approximately in OLS form. Measuring the return of discrete categories is equivalent to the use of a dummy variable in a regression; applying a filter rule (a rule such that investment takes place if certain conditions are met) is equivalent to the use of discrete categories; non-linear relationships can be captured by including powers of a variable in the regression; the approach embraces multiple regression, not just simple regression; Bayesian approaches yield similar results to a weighted sum of regression estimates estimated over different time periods.

Premise 3: Period definition. The *observation period* is the interval between one payment and the next. Each item of data is available at the start of the observation period. Each agent uses this information to determine their demand, a price is set, and the agent takes a position. The security pays a return at the end of the observation return. The *estimation period* is formed from T observation periods. The length T of the estimation period is assumed to be the same for every investor. It is convenient to number estimation periods consecutively into the past - 0 is the current period, 1 is the previous period, 2 the period before that etc. This means that periods into the future take negative indices, -1, -2 etc. Assume $T > 2$. (3)

Premise 4: Data set. Without loss of generality, we assume that each agent who uses data orthogonalizes the variables in their data set. As the estimates of orthogonal regressors are statistically independent of each other under OLS, we can consider the data variables separately rather than look at the full data set for each agent. Define:

$\mathbf{X}_j : T \times 1$, a single variable used by an agent (4)

$\mathbf{X}_{original} : T \times J - 1$, compilation containing $J - 1 \geq 1$ distinct variables (5)

We suppose that the independent variables are fixed, i.e. $\mathbf{X}_{original}$ is the same in every estimation period. This simple assumption can be replaced with a more general statement in terms of probability limits without altering the argument to follow.

Define $\operatorname{rank}\left(\mathbf{X}_{original}\right) = N \le \min\left(T, J-1\right)$ so (6)

$N \ge 1$ by (5) (7)

$N \le T$ by (6) (8)

$N \le J-1$ by (6) (9)

Define $\mathbf{X} : T \times N$, normalized orthogonal dataset forming a basis for the original dataset:

$\mathbf{X}'\mathbf{X} = \mathbf{I}_{N*N}$ orthonormal dataset (10)

$\operatorname{rank}\left(\mathbf{X}\right) = N$ by (6) (11)

We express each variable $\mathbf{X}_j$ as a linear function of this basis **X**:

$\mathbf{X}_j = \mathbf{X}\mathbf{a}_j \ : T \times 1$, data set *j*, where (12)

$\mathbf{a}_j = \left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{X}_j : N \times 1$, *strategy* vector; there are $J-1$ data strategy vectors. (13)

$\mathbf{a}_j{}'\mathbf{a}_j = 1$, data strategy vectors are normalized. (14)

Premise 5: Generation of gross returns. We assume that the gross return in each estimation period per unit of the security is generated according to:

$$\mathbf{y} = \mathbf{X}\boldsymbol{\mu} + \boldsymbol{\upsilon} \tag{15}$$

where $\mathbf{y} : T \times 1$ are the gross returns (or payoffs) per unit of the security, for the T observation periods in an estimation period

$\mathbf{X} : T \times N$, the normalized orthogonal basis of the data series

$\boldsymbol{\mu} \ne \mathbf{0} : N \times 1$, non-zero coefficient of gross return per share, where

$\boldsymbol{\mu}'\boldsymbol{\mu} = 1$, shares can be denominated so that return is normalized (16)

$\boldsymbol{\upsilon} : T \times 1$, disturbance term which is independently distributed, i.e.

$$\operatorname{E}\left[\boldsymbol{\upsilon}\right] = 0 \tag{17}$$

$$\operatorname{E}\left[\boldsymbol{\upsilon}\boldsymbol{\upsilon}'\right] = \sigma^2\mathbf{I}_{T*T} \tag{18}$$

Premise 6: Estimation. We include the price strategy so there are now *J trading strategies*

(19). Each strategy *j* specifies either a data or price equation to model net return $r = y - p$, not gross return *y* given at (15).

Data: $\mathbf{r}_t = \mathbf{X}_j \boldsymbol{\beta}_j + \boldsymbol{\upsilon}_{jt}$ (20)

Price: $\mathbf{r}_t = \mathbf{p}_t \cdot \rho_P + \boldsymbol{\upsilon}_{Pt}$ (21)

where $\mathbf{r}_t = \mathbf{y}_t - \mathbf{p}_t \ : T \times 1$, net return per unit in estimation period *t* (22)

$\mathbf{p}_t : T \times 1$, price of the security for each observation period in an estimation period *t*.

$\boldsymbol{\beta}_j : scalar$, coefficient applying to data set $\mathbf{X}_j$

$\rho_P : scalar$, coefficient of price

$\boldsymbol{\upsilon}_{jt}, \boldsymbol{\upsilon}_{Pt} : T \times 1$, disturbance term applying to the model of strategy *j* or *P*. Each strategy will have a different disturbance term as each assumes a different underlying model.

We denote the estimation period in which an investor creates an estimate by a *superscript*:

$$\hat{\boldsymbol{\beta}}_j^t = \left(\mathbf{X}_j{}'\mathbf{X}_j\right)^{-1} \mathbf{X}_j{}'\mathbf{r}_t \ : scalar$$ data estimate created in period *t* by strategy *j* (23)

$$\hat{\rho}^t = \left(\mathbf{p}_t{}'\mathbf{p}_t\right)^{-1} \mathbf{p}_t{}' \, \mathbf{r}_t : scalar$$ price estimate created in estimation period *t* (24)

Each investor forms a prediction in the current period 0 for the strategies they are using:

$\hat{\mathbf{r}}_{j0}^t = \mathbf{X}_j \hat{\boldsymbol{\beta}}_j^t = \mathbf{Xa}_j \hat{\boldsymbol{\beta}}_j^t : T \times 1$ predicted net return for strategy *j* estimated in *t* (25)

$\hat{\mathbf{r}}_{P0}^t = \mathbf{p}_0 \cdot \hat{\rho}^t : T \times 1$ predicted net return for price estimated in *t* (26)

Premise 7: Trading demand. We assume that trading demand of the agent in each observation period is proportional to the return expected by the agent given the available data. This demand function is derived by Grossman (1976 pp 574-576) from utility considerations. Grossman calculates the demand for a risky asset of an agent operating in a market with two goods, one risky and one risk-free, and an attitude towards risk described by a CARA (constant absolute risk aversion) utility function. The result is a linear demand function (Grossman 1976 equation 11).

The question arises as to whether agents can reconcile the expected return with their realised returns. We interpret the demand function as a decision making rule ('rule of thumb') rather than a formal statement of expectation. It is only necessary that investment according to this rule is consistent with profitable trading; the expected return given by the rule is not realised as such.

The total return prediction of agent *k* is comprised of predictions for one or more strategies. Since the agent's data strategies are orthogonal by Premise 4, the estimates are statistically independent so predictions for different strategies can be added linearly. As stated above in Premise 2, a price regression is carried out separately to a data regression. We take it that the price prediction is added to the predictions derived from the various data strategies.

$$\hat{\mathbf{r}}_k = \hat{\mathbf{r}}_{S1} + \hat{\mathbf{r}}_{S2} + \hat{\mathbf{r}}_P \text{, where} \tag{27}$$

$\hat{\mathbf{r}}_k : T \times 1$ is the total predicted return for agent *k*

S1, S2… denote the various strategies

The trader then transacts the security in amount given by

$$Q_k = T_k \ \hat{\mathbf{r}}_k \text{, where} \tag{28}$$

$Q_k : T \times 1$, number of units of the security demanded by agent *k* in the current estimation period 0, or supplied if it is negative.

$T_k > 0 : scalar$, absolute weight of money: units of security demanded by agent *k* per unit of expected return. The absolute weight of money is always positive.

If the expected return $\hat{\mathbf{r}}_k$ is positive then the quantity $T_k \ \hat{\mathbf{r}}_k$ will be positive, and represents demand for the security. If the expected return is negative then the quantity $T_k \ \hat{\mathbf{r}}_k$ will be negative, and represent supply of the security. Observe that agent *k*'s demand for each strategy can be calculated separately:

$$Q_k = T_k \left(\hat{\mathbf{r}}_{S1} + \hat{\mathbf{r}}_{S2} + \hat{\mathbf{r}}_P \right) = T_k \hat{\mathbf{r}}_{S1} + T_k \hat{\mathbf{r}}_{S2} + T_k \hat{\mathbf{r}}_P \qquad \text{by (28), (27)} \tag{29}$$

so if we consider the total demand for the security, we see that demand can be considered by strategy as well as by agent:

$$\begin{array}{lllllll} & & Strategy\,1 & Strategy\,2 & Strategy\,3 & Strategy\,P & \\ Agent\,1 & Q_{A1} = & T_{A1}\,\hat{\mathbf{r}}_{S1} & T_{A1}\,\hat{\mathbf{r}}_{S2} & & T_{A1}\,\hat{\mathbf{r}}_{P} & \\ Agent\,2 & Q_{A2} = & & T_{A2}\,\hat{\mathbf{r}}_{S2} & T_{A2}\,\hat{\mathbf{r}}_{S3} & T_{A2}\,\hat{\mathbf{r}}_{P} & (30) \\ Agent\,3 & Q_{A3} = & & T_{A3}\,\hat{\mathbf{r}}_{S2} & & & \\ Total & & \overline{T_{A1}\,\hat{\mathbf{r}}_{S1}} & \overline{\left(T_{A1}+T_{A2}+T_{A3}\right)\hat{\mathbf{r}}_{S2}} & \overline{T_{A2}\,\hat{\mathbf{r}}_{S3}} & \overline{\left(T_{A1}+T_{A2}\right)\hat{\mathbf{r}}_{P}} & \end{array}$$

For the development which follows, we require the weight of money by strategy because estimates are made and updated for the strategies, not the agents. Define:

$T_j = \sum_k T_{kj} > 0 : scalar$, the absolute weight of money applying to strategy *j*, (31)

made up of agent amounts T_{kj} which for each agent may be zero or positive according to whether the agent uses that strategy or not.

$T_{TR} = \sum_j T_j > 0 : scalar$, total absolute weight of money for traders. (32)

$T_{TR} \geq \sum_k T_k$ because weight of money T_k can apply to more than one strategy. (33)

$Q_j = T_j \hat{\mathbf{r}}_j : T \times 1$, absolute number of units of the security demanded (34)

(or supplied) in the current estimation period 0 by agents using strategy *j*.

Relative weight of money: It is easier to analyze relative weight of money than absolute:

$b_j = \dfrac{T_j}{T_{TR}} > 0 : scalar$, relative '*weight of money*' or '*market share*' of strategy *j* (35)

$b_X = \sum_{j \in X} b_j : scalar$, data strategy weight of money (36)

$b_P : scalar$, price strategy weight of money (37)

$\sum_j b_j = 1$ by (32) (38)

$b_X + b_P = 1$ by (32) (39)

Demand function: Decompose demand according to the period in which coefficient $\hat{\boldsymbol{\beta}}_j$ was estimated. This decomposition is important in understanding how the market moves to equilibrium. Use period weights b_j^t to define a mean value of the regression coefficients.

$$b_j^t : scalar \text{, weight of money of strategy } j \text{ applied to estimates made in period } t \quad (40)$$

$$b_j = \sum_{t=1}^{\infty} b_j^t \quad \text{as defined at (35)} \quad (41)$$

$$b_j \hat{\boldsymbol{\beta}}_{j0} = \sum_{t=1}^{\infty} b_j^t \hat{\boldsymbol{\beta}}_j^t : scalar \text{, average value of data estimates weighted by } b_j^t \quad (42)$$

$$b_P \hat{\rho}_0 = \sum_{t=1}^{\infty} b_P^t \hat{\rho}^t : scalar \text{, average value of price coefficients weighted by } b_P^t \quad (43)$$

We can now use these average estimate values to define the demand from the strategy:

$$\mathbf{Q}_j^t = b_j^t \hat{\mathbf{r}}_j^t : T \times 1 \text{, relative demand by } j \text{ using period } t \text{ estimates, compare (34)} \quad (44)$$

$$\mathbf{Q}_{j0} = \sum_{t=1}^{\infty} b_j^t \hat{\mathbf{r}}_j^t = \sum_{t=1}^{\infty} b_j^t \mathbf{X} \mathbf{a}_j \hat{\boldsymbol{\beta}}_j^t = b_j \mathbf{X} \mathbf{a}_j \hat{\boldsymbol{\beta}}_{j0} \qquad \text{by (44); (25); (42)} \quad (45)$$

$$\mathbf{Q}_{P0} = \sum_{t=1}^{\infty} b_P^t \hat{\mathbf{r}}_P^t = \sum_{t=1}^{\infty} b_j^t \mathbf{p}_0 \hat{\rho}^t = b_P \mathbf{p}_0 \hat{\rho}_0 \qquad \text{by (44); (26); (43)} \quad (46)$$

Premise 8: Natural business demand. Some part (which may be zero) of the demand of each of the *K* agents is is independent of expectations of trading return. We will show below that price taking in this way incurs trading losses. For the purposes of this price-taking behaviour, agents do not form an estimate of expected trading return $\hat{\mathbf{r}}_j$, but rather set a limit on the amount which they are prepared to pay in trading losses in an estimation period. Costs are taken to be negative.

$$Average\ loss\ per\ unit\ weight\ of\ money_k \geq C_k^{MIN} \quad \text{where} \quad (47)$$

$$C_k^{MIN} \leq 0 : scalar \text{, minimum acceptable result per } b_N \text{ per estimation period} \quad (48)$$

We assume that price-taking demand is a function of the data in the same way as data trading is. If the constraint (47) is satisfied for at least one agent *k*, total demand is given by:

$$\mathbf{Q}_N = \frac{T_N}{T_{TR}} \mathbf{X}\mathbf{a}_N = b_N \mathbf{X}\mathbf{a}_N \tag{49}$$

where $T_N = \sum_k T_k > 0 : scalar$, total absolute weight of money of natural business (50)

$\mathbf{Q}_N = \sum_k \mathbf{Q}_k : T \times 1$, total relative natural business demand (51)

$\mathbf{a}_N : N \times 1$ unit vector, strategy vector of the natural business agents (52)

$b_N = \sum_k b_k = \frac{T_N}{T_{TR}} > 0 : scalar$, total relative weight of money of natural business (53)

The natural business demand equation (49) can incorporate an error term, but an error term here would eventually be combined with other sources of error into an estimation error term. For simplicity we omit the error term without changing the outworking of the model.

Relationship to other strategies: The data trading strategies can be used in the form $\mathbf{X}_j = \mathbf{X}\mathbf{a}_j$ or $\mathbf{X}_j = -\mathbf{X}\mathbf{a}_j$ without changing the estimates. Without loss of generality, set the sign of each data strategy $\mathbf{a}_j$ so that it is non-negatively correlated with natural business:

$$\mathbf{a}_j' \mathbf{a}_N \geq 0 \tag{54}$$

Premise 9: Investor reestimation. At the end of each estimation period, a small positive proportion *w* of investors using strategy *j* reestimate their model using data from that estimation period. The scheme accommodates investors who reestimate in every period and use a rolling average of their estimates. Such reestimation is done to achieve more accurate estimates, and may be viewed as equivalent to Bayesian updating of estimates. Investors decide to reestimate independently of the age of their existing estimates, so the same proportion *w* is drawn from every age cohort of estimates. It follows:

$$b_j^t = b_j w (1-w)^{t-1} \tag{55}$$

The same scheme is found in Georges (2008).

Premise 10: Security market equilibrium condition. In each observation period there is zero net supply. Each investor offers long or short bids for the security according to their demands (45), (46) and (49). The Walrasian auctioneer sets a market clearing price at which net demand is zero.

$$\sum_{j\in X}\mathbf{Q}_{j0}+\mathbf{Q}_{P0}+\mathbf{Q}_N=\mathbf{0}_{T\times 1} \qquad \text{the 'demand equation'} \qquad (56)$$

3. Equilibrium in the security market: the first order model

3.1 The price equation

We use Premises 1-10 to calculate the market price and investigate its properties.

Definition: Vector quantities for data strategies. From (45) we see that regression coefficients $\hat{\boldsymbol{\beta}}_j$ weighted by $b_j\mathbf{a}_j$ represent a vector version of the quantity demanded, with dataset **X** not included. Variable $\mathbf{q}_{X0}$ for period 0, which embraces estimates from all periods, is denoted using a subscript 0, and $\mathbf{q}_X^t$, pertaining to the estimates created in period *t*, only uses a superscript *t*.

$$\mathbf{q}_{j0}=b_j\mathbf{a}_j\hat{\boldsymbol{\beta}}_{j0}: N\times 1\text{: vector version of quantity (45) which omits } \mathbf{X}. \qquad (57)$$

$$=\sum_{t=1}^{\infty}b_j^t\mathbf{a}_j\hat{\boldsymbol{\beta}}_j^t \qquad \text{by (45)} \qquad (58)$$

$$\mathbf{q}_{X0}=\sum_{j\in X}\mathbf{q}_{j0}: N\times 1\text{, demand at 0 for all data strategies, estimation times} \qquad (59)$$

$$=\sum_{j\in X}b_j\mathbf{a}_j\hat{\boldsymbol{\beta}}_{j0} \qquad \text{by (57). Note that current estimates } \hat{\boldsymbol{\beta}}_j^0 \text{ not included.} \qquad (60)$$

$$=\sum_{j\in X}\sum_{t=1}^{\infty}b_j^t\,\mathbf{a}_j\hat{\boldsymbol{\beta}}_j^t \qquad \text{by (58)} \qquad (61)$$

$$\mathbf{q}_X^1=\sum_{j\in X}b_j\mathbf{a}_j\hat{\boldsymbol{\beta}}_j^1: N\times 1\text{, quantity vector for period 1 estimates alone, for all data.} \qquad (62)$$

Note that this value uses the full weights b_j not period weights b_j^t.

Definition: Vector quantity for natural business.

$$\mathbf{q}_N=b_N\mathbf{a}_N: N\times 1\text{, version of natural business demand (49) omitting data } \mathbf{X}. \qquad (63)$$

Definition: Weighted price coefficients. Similarly we define weighted price coefficients.

$$\rho_0 = b_P \hat{\rho}_0 = \sum_{t=1}^{\infty} b_P^t \, \hat{\rho}^t : scalar \text{ , weighted price coefficient at 0 using (43)} \tag{64}$$

$$\rho^1 = b_P \hat{\rho}^1 : scalar \text{ , price coefficient estimated in period 1} \tag{65}$$

Note that this value uses the full weight b_P , not period weights b_P^t , as (62) above.

$$\rho = \mathrm{E}\left[\rho_0\right] : scalar \text{ , expected value of the net price coefficient} \tag{66}$$

Preamble: We use demand equation (56) to find a price *p* in each observation period, and vector price $\mathbf{p} : T \times 1$ for the estimation period. We then demonstrate that this vector price **p** can be represented by a unique vector $\boldsymbol{\pi} : N \times 1$. The demand equation in $\mathbb{R}^T$ space yields an equivalent equation in $\mathbb{R}^N$ space referred to as the 'price equation'.

Result 3.1: Price theorem.

(i) *There is a unique vector* $\mathbf{p}_0$ *which gives the market price of the security in each observation period:*

$$\mathbf{p}_0 = -\mathbf{X}\left(\frac{\mathbf{q}_{X0} + b_N \mathbf{a}_N}{\rho_0}\right) \tag{67}$$

(ii) $\mathbf{p}_0 = \mathbf{X}\boldsymbol{\pi}_0$ *has a unique solution* $\boldsymbol{\pi}_0$ *:* (68)

$$\boldsymbol{\pi}_0 = -\frac{\mathbf{q}_{X0} + b_N \mathbf{a}_N}{\rho_0} : N \times 1, \textit{ the 'price equation'} \tag{69}$$

Proof: (i) Apply demand equation (56) to the expressions for predicted net return.

$$\mathbf{0}_{T\times 1} = \sum_{j\in X} \mathbf{Q}_{j0} + \mathbf{Q}_{P0} + \mathbf{Q}_N \quad \text{by (56)} \tag{70}$$

so $$\mathbf{0}_{T\times 1} = \sum_{j} b_j \mathbf{X}\mathbf{a}_j \hat{\boldsymbol{\beta}}_{j0} + b_P \mathbf{p}_0 \hat{\rho}_0 + b_N \mathbf{X}\mathbf{a}_N \quad \text{by (45), (46), (49)} \tag{71}$$

$$\mathbf{p}_0 = -\mathbf{X}\left(\frac{\sum_j b_j \mathbf{a}_j \hat{\boldsymbol{\beta}}_{j0} + b_N \mathbf{a}_N}{b_P \hat{\rho}_0}\right) = -\mathbf{X}\left(\frac{\mathbf{q}_{X0} + b_N \mathbf{a}_N}{\rho_0}\right) \quad \text{by (61), (64)} \tag{72}$$

(ii) $$\mathrm{rank}\left(\mathbf{X}\right) = \dim\left(\mathbf{X}\right) = N \quad \text{by (11)} \tag{73}$$

$$\dim(domain) = \dim(\boldsymbol{\pi}_0) = N \quad \text{by (69)} \tag{74}$$

$$N + \dim(\text{Ker}) = N \quad \text{by rank-nullity theorem so solution to (68) is unique.} \quad \# \tag{75}$$

Definition: Vector quantity for the price strategy. We can now define vector quantities for the price strategy in the same way as the data strategies, by omitting dataset **X**.

$$\mathbf{q}_{P0} = b_P \mathbf{X} \boldsymbol{\pi}_0 \hat{\rho}_0 \quad \text{by (46), (68)} \tag{76}$$

$$\mathbf{q}_P = \boldsymbol{\pi}\, \rho_0 = b_P \boldsymbol{\pi}\, \hat{\rho}_0 = \sum_{t=1}^{\infty} b_P^t\, \boldsymbol{\pi}\, \hat{\rho}^t : N \times 1 \quad \text{omit } \mathbf{X}\text{, (64): vector quantity for price} \tag{77}$$

Corollary 3.2: Price equation: vector quantity version. *The price equation (69) can be written in terms of vector quantities as:*

$$\mathbf{q}_{X0} + \mathbf{q}_P + \mathbf{q}_N = \mathbf{0}_{N\times 1} \tag{78}$$

Proof: $\boldsymbol{\pi}_0 = -\dfrac{\mathbf{q}_{X0} + \mathbf{q}_N}{\rho_0}$ by (69), (63). Multiply out and use (77) # (79)

Henceforth net return can be given as:

$$\mathbf{r}_0 = \mathbf{X}\boldsymbol{\mu} - \mathbf{X}\boldsymbol{\pi}_0 + \boldsymbol{\upsilon}_0 \quad \text{using (15), (22), (68)} \tag{80}$$

We see that price can be represented by the vector $\boldsymbol{\pi}_0$, constant for all observations in an estimation period, rather than vector $\mathbf{p}_0$, which takes a different value in each observation period. The price regressor can be represented by a strategy vector, $\boldsymbol{\pi}_0$, and becomes the $J\,th$ strategy vector. Representing price as a strategy vector allows the treatment of non-price and price data to be combined. Unlike data strategy vectors $\mathbf{a}_j$, price is a function of its period, so the corresponding strategy vector $\boldsymbol{\pi}_t$ is also. Henceforth:

$$\mathbf{a}_{jt} : N \times 1 \text{ either data or price strategy matrix (as distinct from } \mathbf{a}_j \text{ data}, \boldsymbol{\pi}_t \text{ price)} \tag{81}$$

3.2 Estimation and coefficient space

Preamble: We show that the estimates which investors form using OLS regression can be expressed in terms of the strategy vectors alone without reference to the data set **X**.

Result 3.3: Estimation theorem. *Estimates created in estimation period t are given by:*

$$\hat{\boldsymbol{\beta}}_j^t = \left(\mathbf{a}_{jt}{}'\mathbf{a}_{jt}\right)^{-1} \mathbf{a}_{jt}{}'\left(\boldsymbol{\mu} - \boldsymbol{\pi}_t + \mathbf{e}_t\right) = \mathbf{a}_{jt}{}'\left(\boldsymbol{\mu} - \boldsymbol{\pi}_t + \mathbf{e}_t\right) \qquad \textit{using } (14) \qquad (82)$$

where $\hat{\boldsymbol{\beta}}_j^t$ *: scalar , regression coefficient estimated by j in period t for either data or price*

$\mathbf{e}_t = \mathbf{X}_t{}'\boldsymbol{\upsilon}_t : N \times 1$, *matrix of transformed disturbances such that:* (83)

$$\mathrm{E}\left[\mathbf{e}\right] = \mathbf{0} \qquad (84)$$

$$\mathrm{E}\left[\mathbf{e}\mathbf{e}'\right] = \sigma^2 \mathbf{I}_{N \times N} \qquad (85)$$

Proof: Apply the standard OLS regression result and simplify given **X** is orthonormal.

$$\hat{\boldsymbol{\beta}}_j^t = \left(\mathbf{X}_j{}'\mathbf{X}_j\right)^{-1} \mathbf{X}_j{}'\mathbf{r}_t \qquad \text{OLS} \qquad (86)$$

$$= \left(\mathbf{a}_{jt}{}'\mathbf{X}'\mathbf{X}\mathbf{a}_{jt}\right)^{-1} \mathbf{a}_{jt}{}'\mathbf{X}'\left(\mathbf{X}\boldsymbol{\mu} - \mathbf{X}\boldsymbol{\pi}_t + \boldsymbol{\upsilon}_t\right) \qquad \text{by (12), (80)} \qquad (87)$$

$$= \left(\mathbf{a}_{jt}{}'\mathbf{a}_{jt}\right)^{-1} \mathbf{a}_{jt}{}'\left(\boldsymbol{\mu} - \boldsymbol{\pi}_t + \mathbf{e}_t\right) \qquad \text{by (10), (83)} \qquad (88)$$

Finally $\mathrm{E}\left[\mathbf{e}_t\mathbf{e}_t{}'\right] = \mathrm{E}\left[\mathbf{X}'\boldsymbol{\upsilon}_t\boldsymbol{\upsilon}_t{}'\mathbf{X}\right] = \sigma^2 \mathbf{I}_{N*N}$ by (83); (18), (10) # (89)

We see that regression of return against data or price is equivalent to regression of the underlying parameters $\boldsymbol{\mu} - \boldsymbol{\pi}$ against strategy matrix $\mathbf{a}_{jt}$. In a sense the investor's problem does not involve data at all but is a test of how well the investor's data coefficients capture the return coefficients $\boldsymbol{\mu} - \boldsymbol{\pi}$. We have abstracted the problem from 'data space' $\mathbb{R}^T$ to 'coefficient space' $\mathbb{R}^N$. Coefficient space is an analytical construct, it does not have any interpretation in terms of investor experience.

Definition: new and old estimates. We divide the estimates for the current period 0 into the 'new' estimates found in the preceding period 1, and the 'old' estimates for all the periods before that, i.e. $t \geq 2$.

Preamble: We wish to break down coefficients into new and old components, as we will show below that price changes can be characterized using this distinction.

Result 3.4: Coefficient update rule. *The average values of coefficients and quantities are* *related to new estimates by the following weighting rule:*

Data: $\hat{\boldsymbol{\beta}}_{j0} = w\hat{\boldsymbol{\beta}}_j^1 + (1-w)\hat{\boldsymbol{\beta}}_{j1}$ *estimates* (90)

$\mathbf{q}_{X0} = w\mathbf{q}_X^1 + (1-w)\mathbf{q}_{X1}$ *vector quantities* (91)

Price: $\hat{\rho}_0 = w\hat{\rho}^1 + (1-w)\hat{\rho}_1$ *estimates* (92)

$\rho_0 = w\rho^1 + (1-w)\rho_1$ *weighted price coefficient by (64)* (93)

Proof: Result follows from (55). Derivation given for data, derivation for price identical.

$$b_j\hat{\boldsymbol{\beta}}_{j0} = \sum_{t=1}^{\infty} b_j w(1-w)^{t-1}\hat{\boldsymbol{\beta}}_j^t \quad \text{apply (55) to (42)} \tag{94}$$

$$\hat{\boldsymbol{\beta}}_{j0} = w(1-w)^0\,\hat{\boldsymbol{\beta}}_j^1 + (1-w)\sum_{t=2}^{\infty} w(1-w)^{t-1}\,\hat{\boldsymbol{\beta}}_j^t \quad \text{separate } t=1\text{, divide by } b_j \tag{95}$$

$$= w\hat{\boldsymbol{\beta}}_j^1 + (1-w)\hat{\boldsymbol{\beta}}_{j1} \quad \text{similarly apply (55) to (43) for the price strategy} \tag{96}$$

Now $$\sum_{j\in X} b_j\mathbf{a}_j\hat{\boldsymbol{\beta}}_{j0} = w\sum_{j\in X} b_j\mathbf{a}_j\hat{\boldsymbol{\beta}}_j^1 + (1-w)\sum_{j\in X} b_j\mathbf{a}_j\hat{\boldsymbol{\beta}}_{j1} \quad \text{multiply by } b_j\mathbf{a}_j\text{, add over } j \tag{97}$$

$$\mathbf{q}_{X0} = w\mathbf{q}_X^1 + (1-w)\mathbf{q}_{X1} \quad \text{by (60), (62)} \quad \# \tag{98}$$

3.3 The price change principle

Definition: **Estimation matrix**. Consider the total quantity demanded by all trading strategies (not natural business) for estimates generated in one period *t*.

$$\mathbf{q}_{TR}^t = \sum_{j\in X,P} b_j\mathbf{a}_{jt}\hat{\boldsymbol{\beta}}_j^t \quad \text{by (62), (65)} \tag{99}$$

$$= \sum_j b_j\mathbf{a}_{jt}\cdot\mathbf{a}_{jt}'\left(\boldsymbol{\mu} - \boldsymbol{\pi}_t + \mathbf{e}_t\right) \quad \text{by (82)} \tag{100}$$

$$= \left(\sum_j b_j\mathbf{a}_{jt}\mathbf{a}_{jt}'\right)\left(\boldsymbol{\mu} - \boldsymbol{\pi}_t + \mathbf{e}_t\right) \tag{101}$$

We see that the LHS factor can be treated as an operator on the return vector for the period. We refer to the LHS factor as the *estimation matrix*, a weighted average of the investor projection matrices for both data and price traders. For reasons of analytical tractability, we use an asymptotically approximate version of the estimation matrix where average price $\bar{\boldsymbol{\pi}}$

is used rather than the price in each period $\boldsymbol{\pi}_t$. Define normalized average price vector **u**:

$$\mathbf{u} = \left(\bar{\boldsymbol{\pi}}'\bar{\boldsymbol{\pi}}\right)^{-1/2}\bar{\boldsymbol{\pi}} : N \times 1 \text{ unit vector, normalized average price vector, i.e.:} \quad (102)$$

$$\mathbf{u}'\mathbf{u} = 1 \quad (103)$$

The estimation matrix is given by:

$$\mathbf{H} = \sum_{j \in X} b_j \mathbf{a}_j \mathbf{a}_j{}' + b_P \bar{\boldsymbol{\pi}}\left(\bar{\boldsymbol{\pi}}'\bar{\boldsymbol{\pi}}\right)^{-1}\bar{\boldsymbol{\pi}}' = \sum_j b_j \mathbf{a}_j \mathbf{a}_j{}' + b_P \mathbf{u}\mathbf{u}' = \mathbf{H}_X + \mathbf{H}_P : N \times N \quad (104)$$

$$\mathbf{H}_X = \sum_{j \in X} b_j \mathbf{a}_j \mathbf{a}_j{}' : N \times N \text{ estimation matrix component due to data strategies} \quad (105)$$

$$\mathbf{H}_P = b_P \mathbf{u}\mathbf{u}' : N \times N \text{ estimation matrix component due to price strategies} \quad (106)$$

$$\mathbf{H}_j = b_j \mathbf{a}_j \mathbf{a}_j{}' : N \times N \text{, estimation matrix component due to data strategy } j \quad (107)$$

Note $\operatorname{tr}(\mathbf{H}) = \sum_{j \in X} b_j \operatorname{tr}\left(\mathbf{a}_j \mathbf{a}_j{}'\right) + b_P \operatorname{tr}\left(\mathbf{u}\mathbf{u}'\right) = \sum_j b_j = 1$ given trace cyclical property (108)

Preamble: The behaviour of the system depends on how the price vector $\boldsymbol{\pi}$ responds to the new estimates which agents create in each estimation period. We show that price moves in the direction of the new estimates. In this way the discrepancy between price and return is constantly being corrected.

Result 3.5: Price change principle. *In a market governed by P1-P10:*

$$\mathbf{d}\boldsymbol{\pi}_0 \approx -\frac{w}{\rho_1}\mathbf{H}\left(\boldsymbol{\mu}_N - \boldsymbol{\pi}_1 + \mathbf{e}_1\right) \quad (109)$$

where $\mathbf{d}\boldsymbol{\pi} = \boldsymbol{\pi}_0 - \boldsymbol{\pi}_1$ (110)

$$\boldsymbol{\mu}_N = \boldsymbol{\mu} + b_N \mathbf{H}^{-1}\mathbf{a}_N : N \times 1, \textit{ augmented return} \quad (111)$$

Proof: We differentiate the price equation with respect to re-estimation coefficient *w* and evaluate at $w = 0$. At this point the old estimates $\boldsymbol{\beta}_1$, ρ_1 are in equilibrium and drop out.

$$\mathbf{0}_{N\times 1} = b_N \mathbf{a}_N + \mathbf{q}_{X0} + \boldsymbol{\pi}\rho_0 \quad \text{restating (69) price equation} \quad (112)$$

$$\mathbf{0}_{N\times 1} = w\left(b_N \mathbf{a}_N + \mathbf{q}_X^1 + \boldsymbol{\pi}\rho^1\right) + (1-w)\left(b_N \mathbf{a}_N + \mathbf{q}_{X1} + \boldsymbol{\pi}\rho_1\right) \text{ by (91), (93)} \quad (113)$$

Differentiate with respect to *w*:

$$\mathbf{0}_{N\times 1} = \left(b_N \mathbf{a}_N + \mathbf{q}_X^1 + \boldsymbol{\pi}\rho^1\right) - \left(b_N \mathbf{a}_N + \mathbf{q}_{X1} + \boldsymbol{\pi}\rho_1\right) + \frac{\mathbf{d}\boldsymbol{\pi}}{dw}\left(w\rho^1 + (1-w)\cdot\rho_1\right) \tag{114}$$

$$\left.\frac{\mathbf{d}\boldsymbol{\pi}}{dw}\right|_{w=0} = -\frac{b_N \mathbf{a}_N + \mathbf{q}_X^1 + \boldsymbol{\pi}_1\rho^1}{\rho_1} \tag{115}$$

as at $w = 0$, $\boldsymbol{\pi} = \boldsymbol{\pi}_1$ so $b_N \mathbf{a}_N + \mathbf{q}_X^1 + \boldsymbol{\pi}\rho_1 = \mathbf{0}_{N*1}$. Take first term of Taylor expansion: (116)

$$\mathbf{d}\boldsymbol{\pi} \approx -\frac{b_N \mathbf{a}_N + \mathbf{q}_X^1 + \boldsymbol{\pi}_1\rho^1}{\rho_1}\cdot w \qquad \text{i.e. } \left.\frac{\mathbf{d}\boldsymbol{\pi}}{dw}\right|_{w=0}\cdot w \tag{117}$$

$$= -\frac{w}{\rho_1}\left(b_N \mathbf{a}_N + \sum_j b_j \mathbf{a}_j \hat{\boldsymbol{\beta}}_j^1 + b_P \boldsymbol{\pi}_1 \hat{\rho}^1\right) \qquad \text{by (62), (65)} \tag{118}$$

$$= -\frac{w}{\rho_1}\left(b_N \mathbf{a}_N + \sum_{j\in X,P} b_j \mathbf{a}_{j1}\cdot \mathbf{a}_{j1}'\left(\boldsymbol{\mu} - \boldsymbol{\pi}_1 + \mathbf{e}_1\right)\right) \qquad \text{combine } X, P \text{ by (82)} \tag{119}$$

$$\doteq -\frac{w}{\rho_1}\left(b_N \mathbf{a}_N + \mathbf{H}\left(\boldsymbol{\mu} - \boldsymbol{\pi}_1 + \mathbf{e}_1\right)\right) \qquad \text{approximating with (104)} \tag{120}$$

$$= -\frac{w}{\rho_1}\mathbf{H}\left(b_N \mathbf{H}^{-1}\mathbf{a}_N + \left(\boldsymbol{\mu} - \boldsymbol{\pi}_1 + \mathbf{e}_1\right)\right) \qquad \mathbf{H} \text{ invertible by (133) below, apply (111)} \,\# \tag{121}$$

Remark: We emphasise what this result does *not* say – that the price change is given by the *change* in estimates as might be expected – but rather, the price change is determined by new estimates, so price $\boldsymbol{\pi}$ is pushed towards augmented return $\boldsymbol{\mu}_N$.

$$\mathbf{d}\boldsymbol{\pi} \neq -\frac{w}{\rho_1}\left(\mathbf{q}_X^1 - \mathbf{q}_{X1}\right) \qquad \text{intuitive but wrong!} \tag{122}$$

$$\mathbf{d}\boldsymbol{\pi} \approx -\frac{w}{\rho_1}\left(b_N \mathbf{a}_N + \mathbf{q}_X^1 + \boldsymbol{\pi}_1\rho^1\right) \qquad \text{correct! Restating (117)} \tag{123}$$

3.4 Properties of the estimation matrix

Definitions: The estimation matrix **H** is real symmetric so has real orthogonal eigenvectors:

$$\mathbf{H} = \boldsymbol{\Lambda}\boldsymbol{\lambda}\boldsymbol{\Lambda}' \tag{124}$$

where $\boldsymbol{\lambda} : N\times N$, real eigenvalues of **H** (125)

$\boldsymbol{\Lambda} : N\times N$, real orthogonal eigenvectors of estimation matrix **H**

$$\boldsymbol{\Lambda}\,\boldsymbol{\Lambda}' = \mathbf{I}_{N\times N} \tag{126}$$

Note $\operatorname{tr}(\boldsymbol{\lambda}) = \operatorname{tr}(\boldsymbol{\Lambda}'\mathbf{H}\boldsymbol{\Lambda}) = 1$ by (108), i.e. the sum of the eigenvalues is unity. (127)

Result 3.6: Positive definite estimation matrix. *Estimation matrix* $\mathbf{H}$ *is positive definite.*

Proof: $\mathbf{X}^{original} = \mathbf{X}[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_{J-1}]$ by (12) (128)

$\operatorname{rank}[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_{J-1}] \geq \operatorname{rank}(\mathbf{X}^{original}) = N$ by (6) (129)

i.e. $[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_{J-1}]$ spans $\mathbb{R}^N$ (130)

For all $\mathbf{x} \in \mathbb{R}^N \neq \mathbf{0}_{N\times 1}$, there exists $\mathbf{a}_j$ such that $\mathbf{x}'\mathbf{a}_j = c_j \neq 0$ (131)

$$\mathbf{x}'\mathbf{H}_t\mathbf{x} = \mathbf{x}'\left(\sum_j b_j \mathbf{a}_j \mathbf{a}_j{}'\right)\mathbf{x} = \sum_j b_j \cdot c_j \cdot c_j > 0 \qquad \# \tag{132}$$

It follows that estimation matrix **H** is of full rank and can be inverted. Thus:

$$\mathbf{H}^{-1} = \boldsymbol{\Lambda}\,\boldsymbol{\lambda}^{-1}\boldsymbol{\Lambda}' \tag{133}$$

$0 < \lambda < 1$ given eigenvalues of a positive definite matrix are positive, (127) (134)

This property distinguishes the estimation matrix, a weighted average of projection matrices, from the projection matrices themselves which are non-negative definite with $\lambda = 0,1$.

Definitions: The convergence process requires a modified version of estimation matrix **H**:

$\dot{\mathbf{H}} = \mathbf{I} + \frac{w}{\rho}\mathbf{H} : N \times N$, convergence matrix (135)

$\dot{\boldsymbol{\lambda}} = \mathbf{I} + \frac{w}{\rho}\boldsymbol{\lambda} : N \times N$, real eigenvalue matrix of convergence matrix $\dot{\mathbf{H}}$ (136)

Note $\dot{\mathbf{H}} = \boldsymbol{\Lambda}\boldsymbol{\Lambda}' + \frac{w}{\rho}\boldsymbol{\Lambda}\,\boldsymbol{\lambda}\,\boldsymbol{\Lambda}' = \boldsymbol{\Lambda}\left(\mathbf{I} + \frac{w}{\rho}\boldsymbol{\lambda}\right)\boldsymbol{\Lambda}' = \boldsymbol{\Lambda}\,\dot{\boldsymbol{\lambda}}\,\boldsymbol{\Lambda}'$ by (126); (136) (137)

3.5 Convergence of price to value

Preamble: The view that market stability is a property which should be implicit in the model presupposes that real world markets are stable and this is not always the case. Rather, we determine a condition for stability by applying the criterion that eigenvalues of the transition matrix must lie inside the complex unit circle. Here this matrix is $\dot{\mathbf{H}}$.

Result 3.7: Stability condition. *Matrix* $\dot{\mathbf{H}}$ *has roots lying inside the unit circle if:*

$$\rho < -\frac{w\lambda^{MAX}}{2} \tag{138}$$

Proof: The condition for stability is that eigenvalues lie in the unit circle:

$$-1 < \dot{\lambda} < 1 \qquad \text{for each eigenvalue } \dot{\lambda} \tag{139}$$

$$-1 < 1 + \frac{w}{\rho}\lambda < 1 \qquad \text{by (136)} \tag{140}$$

If $\rho < 0$: $\rho < -\frac{w\lambda}{2} < 0$ on rearranging. Condition is most restrictive with λ^{MAX}. (141)

If $\rho > 0$: $\rho > -\frac{w\lambda}{2} > 0$ As $w > 0$ by (55), $\lambda > 0$ by (134), there are no solutions. # (142)

Preamble: We can now show that market price $\boldsymbol{\pi}$ converges to adjusted return vector $\boldsymbol{\mu}_N$.

Result 3.8: Efficient market theorem. *In a market governed by P1-P10 and stability condition (138):*

$$\lim_{t \to -\infty} E\left[\boldsymbol{\pi}_n\right] \approx \boldsymbol{\mu} + b_N \mathbf{H}^{-1}\mathbf{a}_N \qquad \text{(recall future periods have negative indices)} \tag{143}$$

Proof: We express the deviation of price from return as a geometrically decreasing series using the price change principle (109).

$$\mathbf{d\pi} = \boldsymbol{\pi}_0 - \boldsymbol{\pi}_1 \qquad \text{i.e. } -\boldsymbol{\pi}_0 = -\boldsymbol{\pi}_1 - \mathbf{d\pi} \tag{144}$$

$$\boldsymbol{\mu}_N - \boldsymbol{\pi}_0 = \boldsymbol{\mu}_N - \boldsymbol{\pi}_1 + \frac{w}{\rho_1}\mathbf{H}\left(\boldsymbol{\mu}_N - \boldsymbol{\pi}_1 + \mathbf{e}_1\right) \qquad \text{add } \boldsymbol{\mu}_N\text{, use (109)} \tag{145}$$

$$\approx \boldsymbol{\mu}_N - \boldsymbol{\pi}_1 + \frac{w}{\rho_1}\mathbf{H}\left(\boldsymbol{\mu}_N - \boldsymbol{\pi}_1 + \mathbf{e}_1\right) \qquad \text{asymptotic approximation} \tag{146}$$

$$\mathrm{E}\left[\boldsymbol{\mu}_N - \boldsymbol{\pi}_0\right] = \mathrm{E}\left[\boldsymbol{\mu}_N - \boldsymbol{\pi}_1\right] + \frac{w}{\rho}\mathbf{H}\cdot\mathrm{E}\left[\boldsymbol{\mu}_N - \boldsymbol{\pi}_1\right] \qquad \text{take expectations} \tag{147}$$

$$= \left(\mathbf{I} + \frac{w}{\rho}\mathbf{H}\right)\mathrm{E}\left[\boldsymbol{\mu}_N - \boldsymbol{\pi}_1\right] \qquad \text{factorizing} \tag{148}$$

$$= \boldsymbol{\Lambda}\dot{\boldsymbol{\lambda}}\boldsymbol{\Lambda}'\mathrm{E}\left[\boldsymbol{\mu}_N - \boldsymbol{\pi}_1\right] \qquad \text{by (137)} \qquad (149)$$

$$= \boldsymbol{\Lambda}\dot{\boldsymbol{\lambda}}^2\boldsymbol{\Lambda}'\mathrm{E}\left[\boldsymbol{\mu}_N - \boldsymbol{\pi}_2\right] \qquad \text{etc. as } \boldsymbol{\Lambda} \text{ is orthogonal. Apply (139), (111)} \qquad \#\ (150)$$

The result leads us to conclude that:

$$\bar{\boldsymbol{\pi}} \approx \boldsymbol{\mu}_N \qquad (151)$$

so $$\mathbf{d}\boldsymbol{\pi}_0 \approx -\frac{w}{\rho_1}\mathbf{H}\left(\bar{\boldsymbol{\pi}} - \boldsymbol{\pi}_1 + \mathbf{e}_1\right) \text{ rewriting price change principle (109) using (151)} \qquad (152)$$

3.6 Depictions of convergence in the coefficient model

The process described above was simulated in an Excel spreadsheet available on request.

The parameters of the simulation are as follows.

$$N = 17;\ \boldsymbol{\mu} = \begin{bmatrix} 0.5 \\ 0.866 \\ . \end{bmatrix};\ \boldsymbol{\pi}_0 = \begin{bmatrix} 1.00 \\ 0.18 \\ . \end{bmatrix},\ \mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \\ . \end{bmatrix},\ \mathbf{a}_2 = \begin{bmatrix} 0 \\ 1 \\ . \end{bmatrix},\ \mathbf{a}_j : \begin{bmatrix} 0 \\ a_{jj} = 1 \\ 0 \end{bmatrix} \qquad (153)$$
$$dw = 0.001;\ \sigma = 0.1,\ b_{j \in X} = 0.478,\ b_P = 0.191,\ b_N = 0$$

Natural business is not in use and there are 17 dimensions in coefficient space with 17 different data strategies. Dots in the vectors indicate 15 dimensions with 0 entries. Return is assumed to be a function of two parameters.

Figure 3.1: Convergence of price to return in the coefficient model.

The price coefficients converge to the corresponding return coefficients. Notice the speed of convergence. Even though $w = 0.001$, only one thousandth of estimates are updated in each estimation period, the system converges within 20 periods.

DIAGRAM DELETED BECAUSE OF 1MB CONFERENCE UPLOAD LIMIT

Figure 3.2: Trajectory of the price vector though coefficient space.

This figure plots the same data as the previous figure as a trajectory in which connected points denote successive estimation periods. When price settles down at equilibrium (points in the centre of the graph), there is an inverse correlation between the price components.

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4. Equilibrium in the security market: the second order model

4.1 Introduction

We have shown that the market can make a price and that the price converges to security return if the price coefficient is sufficiently negative. Here we find expressions for the price coefficient ρ and a more accurate expression for average price. We do this by estimating the regression coefficients and calculating the correlations between the market variables using infinite series. We show that the price coefficient is negative and that average price does not exactly equal adjusted return $\boldsymbol{\mu}_N$.

4.2 Technical characteristics of a regression market

Preamble: We now establish the technical characteristics of the market, namely the values of the price coefficient ρ and the average price $\bar{\boldsymbol{\pi}}$, the price variance-covariance matrix $\boldsymbol{\Sigma}$ and the correlation of price with other variables.

Result 4.1: Price coefficient. *The relationship between price coefficient* ρ *and the mean price regression coefficient* $\hat{\bar{\rho}}$ *is given by:*

$$\rho = b_P \, \hat{\bar{\rho}} \tag{154}$$

where $\hat{\bar{\rho}} = \mathrm{E}\left[\sum_{t=1}^{\infty} \frac{b_P^t \, \hat{\rho}^t}{b_P}\right]$: *scalar , mean price regression coefficient*. (155)

Proof: $\rho = \mathrm{E}\left[\rho_0\right] = \mathrm{E}\left[\sum_{t=1}^{\infty} b_P^t \, \hat{\rho}^t\right] = b_P \cdot \mathrm{E}\left[\sum_{t=1}^{\infty} \frac{b_P^t \, \hat{\rho}^t}{b_P}\right]$ by (66); (64). Apply (155) # (156)

Preamble: In the limit as price variance goes to zero ($\sigma_\pi^2 \to 0$), price approaches return and orthogonal variations in price are reduced. We create an approximation for the price regression coefficient $\hat{\rho}^t$ by substituting average price $\bar{\boldsymbol{\pi}}$ for price $\boldsymbol{\pi}$ as the regressor, subtracting the mean of the new estimator and adding back the correct mean.

Result 4.2: Price regression coefficient approximation.

$$\hat{\rho}_j^t \approx \left(\overline{\boldsymbol{\pi}}'\overline{\boldsymbol{\pi}}\right)^{-1}\overline{\boldsymbol{\pi}}'\left(\overline{\boldsymbol{\pi}}-\boldsymbol{\pi}_t+\mathbf{e}_t\right)+\hat{\bar{\rho}} \tag{157}$$

Proof: $\hat{\rho}_j^t \approx \left(\overline{\boldsymbol{\pi}}'\overline{\boldsymbol{\pi}}\right)^{-1}\overline{\boldsymbol{\pi}}'\left(\boldsymbol{\mu}-\boldsymbol{\pi}_t+\mathbf{e}_t\right)-$ mean of this estimator + correct mean (158)

$$=\left(\overline{\boldsymbol{\pi}}'\overline{\boldsymbol{\pi}}\right)^{-1}\overline{\boldsymbol{\pi}}'\left(\boldsymbol{\mu}-\boldsymbol{\pi}_t+\mathbf{e}_t\right)-\left(\overline{\boldsymbol{\pi}}'\overline{\boldsymbol{\pi}}\right)^{-1}\overline{\boldsymbol{\pi}}'\left(\boldsymbol{\mu}-\overline{\boldsymbol{\pi}}\right)+\hat{\bar{\rho}} \qquad \# \ (159)$$

Preamble: Many results use the price variance-covariance matrix $\boldsymbol{\Sigma}$. Because this matrix is derived from the price change principle, the orthogonal eigenvectors of $\mathbf{H}$ are also the eigenvectors of $\boldsymbol{\Sigma}$. By using these eigenvalues as coordinate axes, we can switch to a frame of reference where the different price components are independent, i.e. not correlated with each other, and work with a diagonal price variance-covariance matrix.

Result 4.3: Price variance. *Variance $\boldsymbol{\Sigma}$ of price $\boldsymbol{\pi}$ around its mean $\overline{\boldsymbol{\pi}}$ is given by :*

$$\boldsymbol{\Sigma} \approx \sigma^2\left(\mathbf{I}-\dot{\mathbf{H}}\right)\left(\mathbf{I}+\dot{\mathbf{H}}\right)^{-1} \tag{160}$$

where $\boldsymbol{\Sigma}=\mathrm{E}\left[\left(\boldsymbol{\pi}-\overline{\boldsymbol{\pi}}\right)\left(\boldsymbol{\pi}-\overline{\boldsymbol{\pi}}\right)'\right]: N\times N$ *, price covariance matrix* (161)

$$\ddot{\mathbf{H}}= \ \mathbf{I}-\dot{\mathbf{H}} \ =-\frac{w}{\rho}\mathbf{H} \ : N\times N \text{ , complementary matrix to } \dot{\mathbf{H}} \tag{162}$$

Proof: Based on a rearrangement of the price change principle (152).

$$\overline{\boldsymbol{\pi}}-\boldsymbol{\pi}_0=\overline{\boldsymbol{\pi}}-\boldsymbol{\pi}_1+\frac{w}{\rho_1}\mathbf{H}\left(\overline{\boldsymbol{\pi}}-\boldsymbol{\pi}_1+\mathbf{e}_1\right) \qquad \text{from (152), cf. (145)} \tag{163}$$

$$\approx\overline{\boldsymbol{\pi}}-\boldsymbol{\pi}_1+\frac{w}{\rho}\mathbf{H}\left(\overline{\boldsymbol{\pi}}-\boldsymbol{\pi}_1\right)\left(1-\dot{\rho}_1\right)+\frac{w}{\rho}\mathbf{H}\,\mathbf{e}_1\left(1-\dot{\rho}_1\right) \tag{164}$$

where $\dot{\rho}_1=\dfrac{\rho_1-\rho}{\rho}$, relative variation of estimated price coefficient (165)

$$=\left(\mathbf{I}+\frac{w}{\rho}\mathbf{H}\right)\left(\overline{\boldsymbol{\pi}}-\boldsymbol{\pi}_1\right)-\dot{\rho}_1\cdot\frac{w}{\rho}\mathbf{H}\left(\overline{\boldsymbol{\pi}}-\boldsymbol{\pi}_1\right)+\left(1-\dot{\rho}_1\right)\frac{w}{\rho}\mathbf{H}\,\mathbf{e}_1 \tag{166}$$

$$=\dot{\mathbf{H}}\left(\overline{\boldsymbol{\pi}}-\boldsymbol{\pi}_1\right)+\dot{\rho}_1\cdot\ddot{\mathbf{H}}\left(\overline{\boldsymbol{\pi}}-\boldsymbol{\pi}_1\right)-\left(1-\dot{\rho}_1\right)\ddot{\mathbf{H}}\,\mathbf{e}_1 \tag{167}$$

Multiply this expression by its transpose, take expectations, cast out the lower order terms:

$$\boldsymbol{\Sigma}=\dot{\mathbf{H}}\,\boldsymbol{\Sigma}\,\dot{\mathbf{H}}+\sigma^2\,\ddot{\mathbf{H}}^2 \tag{168}$$

$$\mathbf{\Sigma}^D = \dot{\boldsymbol{\lambda}}\,\mathbf{\Sigma}^D\dot{\boldsymbol{\lambda}} + \sigma^2\ddot{\boldsymbol{\lambda}}^2 \qquad \boldsymbol{\Lambda}\text{ - coordinates: diagonalized version of (168)} \quad (169)$$

Consider (169) for the off diagonal elements:

$$\Sigma_{ij} - \dot{\lambda}_i\Sigma_{ij}\dot{\lambda}_j = 0, \quad i \neq j \qquad \text{recalling } \ddot{\lambda}_{ij} = 0 \qquad (170)$$

$$0 = \left(1 - \dot{\lambda}_i\dot{\lambda}_j\right)\Sigma_{ij} = \Sigma_{ij} \qquad \text{given } -1 < \dot{\lambda}_i,\ \dot{\lambda}_j < 1 \text{ by (139)} \qquad (171)$$

So $\mathbf{\Sigma}^D$ is diagonal and:

$$\mathbf{\Sigma}^D\left(\mathbf{I} - \dot{\boldsymbol{\lambda}}^2\right) = \sigma^2\left(\mathbf{I} - \dot{\boldsymbol{\lambda}}\right)^2 \qquad \text{(169), (162), diagonal } \mathbf{\Sigma}^D,\ \dot{\boldsymbol{\lambda}} \text{ commutative} \qquad (172)$$

$$\mathbf{\Sigma}^D\left(\mathbf{I} + \dot{\boldsymbol{\lambda}}\right) = \sigma^2\left(\mathbf{I} - \dot{\boldsymbol{\lambda}}\right) \qquad \text{multiply by } \left(\mathbf{I} - \dot{\boldsymbol{\lambda}}\right)^{-1}\text{, transform back} \qquad \# \quad (173)$$

Preamble: We calculate the correlations between the variables, which are required to find average price, price coefficient and profits.

Result 4.5: Return cross temporal covariance. *Cross temporal covariance is given by:*

$$\mathrm{E}\left[\sum_{t=1}^{\infty} w(1-w)^{t-1}\left(\bar{\boldsymbol{\pi}} - \boldsymbol{\pi}_0 + \mathbf{e}_0\right)\left(\bar{\boldsymbol{\pi}} - \boldsymbol{\pi}_t + \mathbf{e}_t\right)'\right] = \rho\,\mathbf{H}^{-1}\mathbf{\Sigma} \qquad (174)$$

Proof: We form an infinite series using the price change principle given at (152).

$$\bar{\boldsymbol{\pi}} - \boldsymbol{\pi}_0 = \bar{\boldsymbol{\pi}} - \boldsymbol{\pi}_1 + \frac{w}{\rho}\mathbf{H}\left(\bar{\boldsymbol{\pi}} - \boldsymbol{\pi}_1 + \mathbf{e}_1\right) \qquad \text{by (163), use constant } \rho \qquad (175)$$

$$= \dot{\mathbf{H}}\left(\bar{\boldsymbol{\pi}} - \boldsymbol{\pi}_1\right) - \ddot{\mathbf{H}}\mathbf{e}_1 \qquad \text{on rearranging} \qquad (176)$$

$$= \dot{\mathbf{H}}\left(\dot{\mathbf{H}}\left(\bar{\boldsymbol{\pi}} - \boldsymbol{\pi}_2\right) - \ddot{\mathbf{H}}\mathbf{e}_2\right) - \ddot{\mathbf{H}}\mathbf{e}_1 \qquad \text{iterating} \qquad (177)$$

$$= -\ddot{\mathbf{H}}\mathbf{e}_1 - \dot{\mathbf{H}}\,\ddot{\mathbf{H}}\mathbf{e}_2 \ldots \quad -\dot{\mathbf{H}}^{t-1}\ddot{\mathbf{H}}\mathbf{e}_t + \dot{\mathbf{H}}^t\left(\bar{\boldsymbol{\pi}} - \boldsymbol{\pi}_t\right) \qquad (178)$$

So $$\mathrm{E}\left[\left(\bar{\boldsymbol{\pi}} - \boldsymbol{\pi}_0 + \mathbf{e}_0\right)\left(\bar{\boldsymbol{\pi}} - \boldsymbol{\pi}_t + \mathbf{e}_t\right)'\right] \qquad (179)$$

$$= \mathrm{E}\left[\left(-\ddot{\mathbf{H}}\mathbf{e}_1 - \dot{\mathbf{H}}\ddot{\mathbf{H}}\mathbf{e}_2 \ldots - \dot{\mathbf{H}}^{t-1}\ddot{\mathbf{H}}\mathbf{e}_t + \dot{\mathbf{H}}^t\left(\bar{\boldsymbol{\pi}} - \boldsymbol{\pi}_t\right)\right)\left(\bar{\boldsymbol{\pi}} - \boldsymbol{\pi}_t + \mathbf{e}_t\right)'\right] \qquad \text{by (178)} \qquad (180)$$

$$= \mathrm{E}\left[\left(-\dot{\mathbf{H}}^{t-1}\ddot{\mathbf{H}}\mathbf{e}_t + \dot{\mathbf{H}}^t\left(\bar{\boldsymbol{\pi}} - \boldsymbol{\pi}_t\right)\right)\left(\bar{\boldsymbol{\pi}} - \boldsymbol{\pi}_t + \mathbf{e}_t\right)'\right] \qquad \text{discarding uncorrelated terms} \quad (181)$$

$$= \mathrm{E}\left[-\dot{\mathbf{H}}^{t-1}\ddot{\mathbf{H}}\mathbf{e}_t\mathbf{e}_t' + \dot{\mathbf{H}}^t\left(\bar{\boldsymbol{\pi}} - \boldsymbol{\pi}_t\right)\left(\bar{\boldsymbol{\pi}} - \boldsymbol{\pi}_t\right)'\right] \qquad \text{expanding and discarding} \qquad (182)$$

$$= -\dot{\mathbf{H}}^{t-1}\ddot{\mathbf{H}}\boldsymbol{\Omega} + \dot{\mathbf{H}}^{t}\boldsymbol{\Sigma} \quad = \dot{\mathbf{H}}^{t-1}\left(\dot{\mathbf{H}}\boldsymbol{\Sigma} - \sigma^2\ddot{\mathbf{H}}\right) \tag{183}$$

Now $$\dot{\mathbf{H}}\boldsymbol{\Sigma} - \sigma^2\ddot{\mathbf{H}} = \dot{\mathbf{H}}\cdot\sigma^2\left(\mathbf{I}-\dot{\mathbf{H}}\right)\left(\mathbf{I}+\dot{\mathbf{H}}\right)^{-1} - \sigma^2\left(\mathbf{I}-\dot{\mathbf{H}}\right) \qquad \text{by (160)} \tag{184}$$

$$= \sigma^2\left(\mathbf{I}-\dot{\mathbf{H}}\right)\left(\dot{\mathbf{H}}\cdot\left(\mathbf{I}+\dot{\mathbf{H}}\right)^{-1} - \left(\mathbf{I}+\dot{\mathbf{H}}\right)\left(\mathbf{I}+\dot{\mathbf{H}}\right)^{-1}\right) \tag{185}$$

$$= \sigma^2\left(\mathbf{I}-\dot{\mathbf{H}}\right)\left(-\left(\mathbf{I}+\dot{\mathbf{H}}\right)^{-1}\right) = -\boldsymbol{\Sigma} \tag{186}$$

so $$\mathrm{E}\left[\left(\bar{\boldsymbol{\pi}} - \boldsymbol{\pi}_0 + \mathbf{e}_0\right)\left(\bar{\boldsymbol{\pi}} - \boldsymbol{\pi}_t + \mathbf{e}_t\right)'\right] = -\dot{\mathbf{H}}^{t-1}\boldsymbol{\Sigma} \qquad \text{substituting (186) in (183)} \tag{187}$$

and $$\mathrm{E}\left[\sum_{t=1}^{\infty} w(1-w)^{t-1}\left(\bar{\boldsymbol{\pi}} - \boldsymbol{\pi}_0 + \mathbf{e}_0\right)\left(\bar{\boldsymbol{\pi}} - \boldsymbol{\pi}_t + \mathbf{e}_t\right)'\right] = -\sum_{t=1}^{\infty} w(1-w)^{t-1}\,\dot{\mathbf{H}}^{t-1}\boldsymbol{\Sigma} \qquad \text{by (187)} \tag{188}$$

$$= -w\cdot\mathbf{I}\cdot\left(\mathbf{I} - (1-w)\dot{\mathbf{H}}\right)^{-1}\boldsymbol{\Sigma} \qquad \text{sum geometric progression} \tag{189}$$

$$= -w\ddot{\mathbf{H}}^{-1}\boldsymbol{\Sigma} \quad = -w\left(-\frac{w}{\rho}\mathbf{H}\right)^{-1}\boldsymbol{\Sigma} \qquad \text{take limit as } w\to 0\text{, use (162)} \quad \#(190)$$

Corollary 4.6: Return – price coefficient covariance. *The correlation between return and the price coefficient* ρ *is given by:*

$$\mathrm{E}\left[\left(\boldsymbol{\pi}_1 - \bar{\boldsymbol{\pi}}\right)\left(\rho_1 - \rho\right)\right] \approx -b_P\rho\,\mathbf{H}^{-1}\boldsymbol{\Sigma}\cdot\bar{\boldsymbol{\pi}}\left(\bar{\boldsymbol{\pi}}'\bar{\boldsymbol{\pi}}\right)^{-1} \approx -b_P\rho\,\mathbf{H}^{-1}\boldsymbol{\Sigma}\,\mathbf{u} \tag{191}$$

Proof: Draw on the previous approximate coefficient (157) and correlation (174) results.

$$\left(\bar{\boldsymbol{\pi}} - \boldsymbol{\pi}_1 + \mathbf{e}_1\right)\left(\rho_1 - \rho\right) = \left(\bar{\boldsymbol{\pi}} - \boldsymbol{\pi}_1 + \mathbf{e}_1\right)\left(\sum_{t=2}^{\infty} b_P w(1-w)^{t-2}\,\hat{\rho}_j^t - \rho\right) \qquad \text{by (64),(55)} \tag{192}$$

$$\approx \left(\bar{\boldsymbol{\pi}} - \boldsymbol{\pi}_1 + \mathbf{e}_1\right)\cdot\left(\sum_{t=2}^{\infty} b_P w(1-w)^{t-2}\left(\left(\bar{\boldsymbol{\pi}}'\bar{\boldsymbol{\pi}}\right)^{-1}\bar{\boldsymbol{\pi}}'\left(\bar{\boldsymbol{\pi}} - \boldsymbol{\pi}_t + \mathbf{e}_t\right) + \hat{\hat{\rho}}\right) - b_P\hat{\hat{\rho}}\right) \text{(157), (154)} \tag{193}$$

$b_P\hat{\hat{\rho}}$ cancels out, and $\left(\bar{\boldsymbol{\pi}}'\bar{\boldsymbol{\pi}}\right)^{-1}\bar{\boldsymbol{\pi}}'$, $\left(\bar{\boldsymbol{\pi}} - \boldsymbol{\pi}_t + \mathbf{e}_t\right)$ can be transposed as it is scalar:

$$= b_P\left(\sum_{t=2}^{\infty} w(1-w)^{t-2}\left(\bar{\boldsymbol{\pi}} - \boldsymbol{\pi}_1 + \mathbf{e}_1\right)\left(\bar{\boldsymbol{\pi}} - \boldsymbol{\pi}_t + \mathbf{e}_t\right)'\cdot\bar{\boldsymbol{\pi}}\left(\bar{\boldsymbol{\pi}}'\bar{\boldsymbol{\pi}}\right)^{-1}\right) \tag{194}$$

so $$\mathrm{E}\left[\left(\bar{\boldsymbol{\pi}} - \boldsymbol{\pi}_1 + \mathbf{e}_1\right)\left(\rho_1 - \rho\right)\right] = b_P\left(\rho\,\mathbf{H}^{-1}\boldsymbol{\Sigma}\right)\cdot\bar{\boldsymbol{\pi}}\left(\bar{\boldsymbol{\pi}}'\bar{\boldsymbol{\pi}}\right)^{-1} \qquad \text{take expectations, use (174)} \tag{195}$$

Now $\mathrm{E}\left[\mathbf{e}_1\left(\rho_1 - \rho\right)\right] = 0$ as $\mathbf{e}_1$ is independent of ρ_1. Reverse sign for result. #(196)

4.3 The expected price equation

Definitions: Divide variance into the components collinear and orthogonal to the unit average price **u**. The component of a vector collinear to the unit average price $\mathbf{u}$, or variance calculated for such a vector, is denoted ||, and components orthogonal to **u** are denoted $\perp$.

$$\sigma_{\pi}^{2} = \operatorname{tr}\left(\boldsymbol{\Sigma}\right) : scalar \text{, total price variance, using (108)} \quad (197)$$

$$\sigma_{\|}^{2} = \mathbf{u}'\,\boldsymbol{\Sigma}\,\mathbf{u} : scalar \text{, collinear price variance} \quad (198)$$

$$\sigma_{\perp}^{2} = \sigma_{\pi}^{2} - \sigma_{\|}^{2} : scalar \text{, orthogonal price variance} \quad (199)$$

$$\sigma_{C}^{2} = \sigma_{\perp}^{2} - \sigma_{\|}^{2} = \sigma_{\pi}^{2} - 2\sigma_{\|}^{2} : scalar \text{, `conjugate' price variance} \quad (200)$$

Conjugate variance is a modified variance obtained by subtracting collinear variance from orthogonal variance.

Price based coordinates: We introduce price-based coordinates, where unit price **u** is represented by a unit vector with unity in the top position. We convert to price-based coordinates by multiplying by $\boldsymbol{\Lambda}^{PRICE}{}'$ on the LHS, where $\boldsymbol{\Lambda}^{PRICE}$ gives the coordinates of a set of price-based axes under the original coordinate system. Note $\boldsymbol{\Lambda}^{PRICE} \neq \boldsymbol{\Lambda}$ in general.

$$\boldsymbol{\pi} = \begin{bmatrix} \pi_{\|} \\ \boldsymbol{\pi}_{\perp} \end{bmatrix} = \begin{bmatrix} 1+y \\ \mathbf{x} \end{bmatrix} \quad \text{where } \operatorname{E}\left[\mathbf{x}\right] = \mathbf{0} \quad \text{so } \overline{\boldsymbol{\pi}} = \begin{bmatrix} 1+\overline{y} \\ \mathbf{0}_{N-1\times 1} \end{bmatrix}, \ \mathbf{u} = \begin{bmatrix} 1 \\ \mathbf{0}_{N-1\times 1} \end{bmatrix} \quad (201)$$

$$\boldsymbol{\mu} = \begin{bmatrix} \sqrt{1-\mathbf{z}'\mathbf{z}} \\ \mathbf{z} \end{bmatrix} \approx \begin{bmatrix} 1-\frac{1}{2}\mathbf{z}'\mathbf{z} \\ \mathbf{z} \end{bmatrix} \quad (202)$$

where $\pi_{\|} = \boldsymbol{\pi}'\mathbf{u} : scalar$, collinear price, component in direction of unit price **u** (203)

$$y = \pi_{\|} - 1 : scalar \text{, offset of collinear price } \pi_{\|} \text{ from unity} \quad (204)$$

$\mathbf{z} : N-1\times 1$, those components of $\boldsymbol{\mu}$ which are not in the direction of unit price **u**

By (16), the first element of $\boldsymbol{\mu}$ is calculated so the norm of the vector is unity.

Result 4.7: price regression approximations. *The mean price coefficient* $\hat{\overline{\rho}}$ *and* *corresponding mean predicted return* $\hat{\bar{\rho}} \cdot \bar{\boldsymbol{\pi}}$ *can be approximated as follows:*

$$\hat{\bar{\rho}} \approx \mathbf{u}'\left(\boldsymbol{\mu} - \bar{\boldsymbol{\pi}}\right) - \sigma_C^2 \tag{205}$$

$$\hat{\bar{\rho}} \cdot \bar{\boldsymbol{\pi}} \approx \mathbf{u}\mathbf{u}'\left(\boldsymbol{\mu} - \bar{\boldsymbol{\pi}}\right) - \sigma_C^2 \mathbf{u} \tag{206}$$

Proof: We calculate out a price regression using price based coordinates and simplify the results by eliminating higher order terms. We then compare these results with the expressions given above.

$$\left(\boldsymbol{\pi}'\boldsymbol{\pi}\right)^{-1} \boldsymbol{\pi}'\boldsymbol{\mu} = \left(\begin{bmatrix} 1+y & \mathbf{x}' \end{bmatrix} \begin{bmatrix} 1+y \\ \mathbf{x} \end{bmatrix}\right)^{-1} \begin{bmatrix} 1+y & \mathbf{x}' \end{bmatrix} \begin{bmatrix} 1-\frac{1}{2}\mathbf{z}'\mathbf{z} \\ \mathbf{z} \end{bmatrix} \tag{207}$$

$$= \frac{(1+y)\left(1-\frac{1}{2}\mathbf{z}'\mathbf{z}\right) + \mathbf{x}'\mathbf{z}}{1+2y+y^2+\mathbf{x}'\mathbf{x}} \qquad \text{expanding} \tag{208}$$

Use this expression to calculate the values of variables $\hat{\bar{\rho}}, \hat{\bar{\rho}}\,\bar{\boldsymbol{\pi}}$ in these coordinates:

$$\hat{\bar{\rho}} = \mathrm{E}\left[\left(\boldsymbol{\pi}_t'\boldsymbol{\pi}_t\right)^{-1} \boldsymbol{\pi}_t'\left(\boldsymbol{\mu} - \boldsymbol{\pi}_t + \mathbf{e}_t\right)\right] \tag{209}$$

$$= \mathrm{E}\left[\frac{(1+y)\left(1-\frac{1}{2}\mathbf{z}'\mathbf{z}\right) + \mathbf{x}'\mathbf{z}}{1+2y+y^2+\mathbf{x}'\mathbf{x}} - 1\right] \qquad \text{by (208), given } \boldsymbol{\pi}_t,\ \mathbf{e}_t \text{ are uncorrelated} \tag{210}$$

We evaluate the two terms in the numerator separately:

1st: $$\frac{\left(1-\frac{1}{2}\mathbf{z}'\mathbf{z}\right)(1+y)}{1+2y+y^2+\mathbf{x}'\mathbf{x}} \approx \left(1-\frac{1}{2}\mathbf{z}'\mathbf{z}\right)(1+y)\left(1-\left(2y+y^2+\mathbf{x}'\mathbf{x}\right)+\left(2y+y^2+\mathbf{x}'\mathbf{x}\right)^2\right) \tag{211}$$

$$\approx 1 - y + y^2 - \mathbf{x}'\mathbf{x} - \frac{1}{2}\mathbf{z}'\mathbf{z} \qquad \text{expand, suppress 3rd order terms} \tag{212}$$

2nd: $$\mathrm{E}\left[\left(\frac{\mathbf{x}'}{1+2y+y^2+\mathbf{x}'\mathbf{x}}\right)\mathbf{z}\right] = 0 \qquad \text{as left factor } \mathrm{f}\left(-\mathbf{x}\right) = -\mathrm{f}\left(\mathbf{x}\right) \tag{213}$$

So $$\hat{\bar{\rho}} \approx \mathrm{E}\left[1 - y + y^2 - \mathbf{x}'\mathbf{x} - \frac{1}{2}\mathbf{z}'\mathbf{z} - 1\right] \qquad \text{substituting (212), (213) in (210)} \tag{214}$$

$$= \mathrm{E}\left[-y + \bar{y}^2 + \left(y - \bar{y}\right)^2 - \mathbf{x}'\mathbf{x} - \frac{1}{2}\mathbf{z}'\mathbf{z}\right] \tag{215}$$

$$= -\bar{y} + \bar{y}^2 - \sigma_C^2 - \frac{1}{2}\mathbf{z}'\mathbf{z} \qquad \text{by (198), (199), (200)} \tag{216}$$

and $$\hat{\rho}\,\bar{\boldsymbol{\pi}} = \bar{\boldsymbol{\pi}} \cdot \mathrm{E}\left[\left(\boldsymbol{\pi}'\boldsymbol{\pi}\right)^{-1}\boldsymbol{\pi}'\left(\boldsymbol{\mu} - \boldsymbol{\pi} + \mathbf{e}\right)\right] \tag{217}$$

$$= \begin{bmatrix} 1+\bar{y} \\ \mathbf{0} \end{bmatrix}\left(-\bar{y} + \bar{y}^2 - \sigma_C^2 - \tfrac{1}{2}\mathbf{z}'\mathbf{z}\right) \qquad \text{applying (216)} \tag{218}$$

$$\approx \begin{bmatrix} -\bar{y} - \sigma_C^2 - \tfrac{1}{2}\mathbf{z}'\mathbf{z} \\ \mathbf{0} \end{bmatrix} \qquad \text{suppressing third order terms} \tag{219}$$

Calculate the approximate expressions and compare:

$$\mathbf{u}'\left(\boldsymbol{\mu} - \bar{\boldsymbol{\pi}}\right) - \sigma_C^2 = \begin{bmatrix} 1 & \mathbf{0} \end{bmatrix}\left(\begin{bmatrix} 1 - \tfrac{1}{2}\mathbf{z}'\mathbf{z} \\ \mathbf{z} \end{bmatrix} - \begin{bmatrix} 1+\bar{y} \\ \mathbf{0} \end{bmatrix}\right) - \sigma_C^2 \tag{220}$$

$$= -\bar{y} - \sigma_C^2 - \tfrac{1}{2}\mathbf{z}'\mathbf{z} \qquad \text{as (216) but for } \bar{y}^2 \tag{221}$$

We see below that $\bar{y}$ is the same order of magnitude as σ_C^2, $\bar{y}^2$ is lower order of magnitude.

$$\mathbf{u}\mathbf{u}'\left(\boldsymbol{\mu} - \bar{\boldsymbol{\pi}}\right) - \sigma_C^2\mathbf{u} = \begin{bmatrix} 1 \\ \mathbf{0} \end{bmatrix}\begin{bmatrix} 1 & \mathbf{0} \end{bmatrix}\left(\begin{bmatrix} 1 - \tfrac{1}{2}\mathbf{z}'\mathbf{z} \\ \mathbf{z} \end{bmatrix} - \begin{bmatrix} 1+\bar{y} \\ \mathbf{0} \end{bmatrix}\right) - \sigma_C^2\begin{bmatrix} 1 \\ \mathbf{0} \end{bmatrix} \tag{222}$$

$$= \begin{bmatrix} -\bar{y} - \sigma_C^2 - \tfrac{1}{2}\mathbf{z}'\mathbf{z} \\ \mathbf{0} \end{bmatrix} \qquad \text{as (219)} \qquad \# \tag{223}$$

Definitions: Define the following components of price strategy demand:

$$\mathbf{q}_{PD} = -b_P\sigma_C^2\mathbf{u} : N \times 1, \text{ price distortion component of price quantity } \mathrm{E}\left[\rho_0 \cdot \boldsymbol{\pi}_0\right] \tag{224}$$

$$\mathbf{q}_C = \mathrm{E}\left[\left(\rho_0 - \rho\right)\cdot\left(\boldsymbol{\pi}_0 - \bar{\boldsymbol{\pi}}\right)\right] : N \times 1, \text{ price-price coefficient correlation} \tag{225}$$

$$= -b_P\rho\,\mathbf{H}^{-1}\boldsymbol{\Sigma}\,\mathbf{u} \qquad \text{by (191)} \tag{226}$$

Evaluation of the $\mathbf{q}_C$ term suggests that it is not significant relative to other terms, so take:

$$\mathbf{q}_C \doteq \mathbf{0}_{N\times 1} \tag{227}$$

Preamble: We now determine average price. While the calculations are straightforward, they call on all of the previous results in this section. By (143), price converges to augmented return $\boldsymbol{\mu}_N$, so we might expect to find:

$$\bar{\boldsymbol{\pi}} = \boldsymbol{\mu} + \mathbf{H}^{-1}\mathbf{q}_N \qquad \text{rearrange (111) using (63)} \tag{228}$$

However average price is distorted by the bias σ_C^2 in the estimation of the price coefficient.

Result 4.8: Expected price equation. *In a market governed by P1-P10, (138):*

Expected price:

$$\mathbf{H}(\boldsymbol{\mu}-\overline{\boldsymbol{\pi}})+\mathbf{q}_N+\mathbf{q}_{PD}\approx\mathbf{0}_{N\times1} \tag{229}$$

$$\mathbf{H}(\boldsymbol{\mu}-\overline{\boldsymbol{\pi}})+b_N\mathbf{a}_N-b_P\sigma_C^2\mathbf{u}\approx\mathbf{0}_{N\times1} \quad \textit{applying (63), (224)} \tag{230}$$

$$\overline{\boldsymbol{\pi}}\approx\boldsymbol{\mu}+b_N\mathbf{H}^{-1}\mathbf{a}_N-b_P\sigma_C^2\mathbf{H}^{-1}\mathbf{u} \quad \textit{applying (63), (224)} \tag{231}$$

Price equation components:

$$\mathrm{E}[\mathbf{q}_{X0}]=\mathbf{H}_X(\boldsymbol{\mu}-\overline{\boldsymbol{\pi}}) \tag{232}$$

$$\mathrm{E}[\rho_0\boldsymbol{\pi}_0]=\rho\,\mathbf{u} \tag{233}$$

$$\mathrm{E}[\rho_0\boldsymbol{\pi}_0]=\mathbf{H}_P(\boldsymbol{\mu}-\overline{\boldsymbol{\pi}})+\mathbf{q}_{PD} \tag{234}$$

Price equation versions:

$$\mathrm{E}[\mathbf{q}_{X0}]+\mathrm{E}[\rho_0\boldsymbol{\pi}_0]+b_N\mathbf{a}_N=\mathbf{0}_{N\times1} \tag{235}$$

$$\sum_{j\in X}b_j\mathbf{a}_j\mathbf{a}_j{}'(\boldsymbol{\mu}-\overline{\boldsymbol{\pi}})+\rho\,\mathbf{u}+b_N\mathbf{a}_N=\mathbf{0}_{N\times1} \quad \textit{by (235), (232), (105), (233)} \tag{236}$$

Proof: Evaluate price and data expected quantities, solve for $\overline{\boldsymbol{\pi}}$ using price equation (69).

$$\mathrm{E}[\rho_0\boldsymbol{\pi}_0]=\rho\,\overline{\boldsymbol{\pi}}+\mathrm{E}[(\rho_0-\rho)\cdot(\boldsymbol{\pi}_0-\overline{\boldsymbol{\pi}})] \quad \text{rearrange} \tag{237}$$

$$=b_P\left(\mathbf{uu}'(\boldsymbol{\mu}-\overline{\boldsymbol{\pi}})-\sigma_C^2\mathbf{u}\right)+\mathrm{E}[(\rho_0-\rho)\cdot(\boldsymbol{\pi}_0-\overline{\boldsymbol{\pi}})] \quad \text{by (154), (206)} \tag{238}$$

$$\doteq\mathbf{H}_P(\boldsymbol{\mu}-\overline{\boldsymbol{\pi}})+\mathbf{q}_{PD}\quad+0 \quad \text{by (106), (224), (225), (227)} \tag{239}$$

further $$\mathrm{E}[\rho_0\boldsymbol{\pi}_0]=b_P\left(\mathbf{uu}'(\boldsymbol{\mu}-\overline{\boldsymbol{\pi}})-\sigma_C^2\mathbf{u}\right) \quad \text{returning to (238), use (227)} \tag{240}$$

$$=b_P\mathbf{u}\left(\mathbf{u}'(\boldsymbol{\mu}-\overline{\boldsymbol{\pi}})-\sigma_C^2\right) \quad \text{factor} \tag{241}$$

$$=b_P\mathbf{u}\,\hat{\overline{\rho}}=\rho\,\mathbf{u} \quad \text{by (205); (154)} \tag{242}$$

now $$\mathbf{q}_{X0}=\sum_{j\in X}\sum_{t=1}^{\infty}b_j^t\,\mathbf{a}_j\mathbf{a}_j{}'(\boldsymbol{\mu}-\boldsymbol{\pi}_t+\mathbf{e}_t) \quad \text{by (61), (82)} \tag{243}$$

$$\mathrm{E}\left[\mathbf{q}_{X0}\right] = \sum_{j\in X}\sum_{t} b_j^t\,\mathbf{a}_j\mathbf{a}_j{}'\mathrm{E}\left[\left(\boldsymbol{\mu}-\boldsymbol{\pi}_t+\mathbf{e}_t\right)\right] \qquad \text{take expectations} \qquad (244)$$

$$= \mathbf{H}_X\left(\boldsymbol{\mu}-\bar{\boldsymbol{\pi}}\right) \qquad \text{by (105)} \qquad (245)$$

now $$\mathrm{E}\left[\mathbf{q}_{X0}\right]+\mathrm{E}\left[\boldsymbol{\pi}_0\rho_0\right]+\mathrm{E}\left[b_N\mathbf{a}_N\right] = \mathbf{0}_{N\times 1} \qquad \text{take expectations of (69)} \qquad (246)$$

$$\mathbf{H}_X\left(\boldsymbol{\mu}-\bar{\boldsymbol{\pi}}\right)+\mathbf{H}_P\left(\boldsymbol{\mu}-\bar{\boldsymbol{\pi}}\right)+\mathbf{q}_{PD}+\mathbf{q}_N = \mathbf{0}_{N\times 1} \qquad \text{by (245), (239). Use (104)} \quad \#(247)$$

4.4. The price coefficient

Definitions: Estimation matrix summary measures. Define the following summary measures of the estimation matrix **H**. The price weight of money can be written as:

$$b_P = \mathbf{u}'\left(b_P\,\mathbf{u}\cdot\mathbf{u}'\right)\mathbf{u} = \mathbf{u}'\,\mathbf{H}_P\,\mathbf{u} \qquad \text{using (103); (106), and in like fashion:} \qquad (248)$$

$$b_T = \left(\mathbf{u}'\mathbf{H}^{-1}\mathbf{u}\right)^{-1} : \textit{scalar}\text{, total weight of money collinear with price} \qquad (249)$$

$$b_{XP} = b_T - b_P : \textit{scalar}\text{, data weight of money collinear with price } (\neq b_X \ (36)) \qquad (250)$$

Result 4.9: price coefficient theorem. *In a market governed by P1-P10, S1-S2:*

$$\rho = -\frac{b_P b_{XP}}{b_T}\sigma_C^2 - b_P b_N\left(\mathbf{a}_N{}'\mathbf{H}^{-1}\mathbf{u}\right) \qquad (251)$$

Proof: Substitute into price regression result (205) using expression (231) for $\boldsymbol{\mu}-\bar{\boldsymbol{\pi}}$:

$$\rho = b_P\left(\mathbf{u}'\left(-\mathbf{H}^{-1}\left(-b_P\sigma_C^2\mathbf{u}+b_N\mathbf{a}_N\right)\right)-\sigma_C^2\right) \qquad (252)$$

$$= b_P\left(b_P\sigma_C^2\mathbf{u}'\mathbf{H}^{-1}\mathbf{u} - b_N\mathbf{u}'\mathbf{H}^{-1}\mathbf{a}_N - \sigma_C^2\right) \qquad \text{expanding} \qquad (253)$$

$$= b_P\left(\frac{b_P}{b_T}\sigma_C^2 - b_N\mathbf{u}'\mathbf{H}^{-1}\mathbf{a}_N - \sigma_C^2\right) \qquad \text{by (249); use (250) for result} \qquad \#(254)$$

In expression (251), price coefficient ρ is expressed in terms of conjugate variance σ_C^2. The conjugate variance is itself a function of price variance matrix $\boldsymbol{\Sigma}$ given at (160), and therefore a function of ρ. The simultaneous equations (251), (160) are non-linear and cannot readily be solved to yield explicit solutions for the variables ρ and expected price $\bar{\boldsymbol{\pi}}$. We can iterate to a solution starting from the approximation that $\bar{\boldsymbol{\pi}} = \boldsymbol{\mu}$.

Stability condition restated: Stability requires that price coefficient ρ is negative by (138):

$$\rho = -\frac{b_P b_{XP}}{b_T}\sigma_C^2 - b_P b_N \left(\mathbf{a}_N{}'\mathbf{H}^{-1}\mathbf{u}\right) < 0 \qquad \text{given (251)} \qquad (255)$$

We find in simulation that the first term is close to zero and the second term dominates. We can restate this condition as:

Stability condition 2: In practical terms, market stability requires that:

$$\mathbf{a}_N{}'\mathbf{H}^{-1}\mathbf{u} > 0 \qquad (256)$$

This is an interesting and testable implication of the model.

5. Equilibrium in the information market

We suppose that we have established the foundations necessary for an investigation of market economics, and to this task we turn. This section examines the economics of the information market, as distinct from the security market. The security market model, or *financial* model, concerns whether the market can make a price and how efficient it is. The information market model, or *economic* model, deal with the market for information on the security. Information must be generated before anyone can trade. We look at how profitable the different types of information are (data or price) in relation to the costs associated with obtaining that information. Information market, or economic, equilibrium occurs where traders are making zero profit (where a component for normal profits is included in cost).

We note that we are drawing a distinction between the financial and economic models which is by no means universal in the literature. In the Kyle (1985) class of literature, the two markets are conflated, and the assumption of perfect competition by ‘market makers’ in the economic market is used to determine price in the security market. By separating the markets as we have done here, the operation of each can be understood separately and perhaps more clearly.

5.1 Profit calculus

Preamble: We exploit the techniques developed above to evaluate profit for each type of trading strategy. We show that profit, like other market variables, can be expressed in terms of coefficient space variables.

Definitions: Trading profit and net profit. Trading profit is the product of quantity and realized return. The inner product of quantity and realized return in coefficient space gives the total trading profit in an estimation period, i.e. trading profit summed over all the observation periods. Realized return $\mathbf{r}_0$ can refer to different components of total return.

$$\Pi_j^{TR} = \mathbf{Q}_j{}' \mathbf{r}_0 : scalar \text{, trading profit of agent } j \tag{257}$$

$$\mathbf{r}_0^{TR} = \mathbf{X}(\boldsymbol{\mu} - \boldsymbol{\pi}_0 + \mathbf{e}_0) : N \times 1 \text{, total return in the period} \tag{258}$$

$$\mathbf{r}_0^{LR} = \mathbf{X}(\boldsymbol{\mu} - \bar{\boldsymbol{\pi}}) : N \times 1 \text{ , long run return generated by fixed component } \boldsymbol{\mu} - \bar{\boldsymbol{\pi}} \tag{259}$$

$$\mathbf{r}_0^{SR} = \mathbf{X}(\bar{\boldsymbol{\pi}} - \boldsymbol{\pi}_0 + \mathbf{e}_0) : N \times 1 \text{, short run return from variable comp. } \bar{\boldsymbol{\pi}} - \boldsymbol{\pi}_0 + \mathbf{e}_0 \tag{260}$$

Result 5.1: Zero trading profit. *The aggregate trading profit, long run profit and short run profit across strategies is zero.*

Proof: $\sum_{j \in X,P,N} \Pi_j = \sum_j \mathbf{Q}_j{}' \mathbf{r}_0 = 0$ by (56), $\mathbf{r}_0$ same for all *j*. Use $\mathbf{r}_0^{TR}, \mathbf{r}_0^{LR}, \mathbf{r}_0^{SR}$. # (261)

Result 5.2: Strategy trading profit. *The trading profit of each type of strategy is given by:*

Natural: $\Pi_N^{TR} = b_N \mathbf{a}_N{}' (\boldsymbol{\mu} - \boldsymbol{\pi}_0 + \mathbf{e}_0)$ (262)

Data: $\Pi_j^{TRt} = b_j^t (\boldsymbol{\mu} - \boldsymbol{\pi}_t + \mathbf{e}_t)' \mathbf{a}_j \mathbf{a}_j{}' (\boldsymbol{\mu} - \boldsymbol{\pi}_0 + \mathbf{e}_0)$ (263)

Price: $\Pi_P^{TR\,t} = b_j^t (\boldsymbol{\mu} - \boldsymbol{\pi}_t + \mathbf{e}_t)' \boldsymbol{\pi}_t \left(\boldsymbol{\pi}_t{}' \boldsymbol{\pi}_t\right)^{-1} \boldsymbol{\pi}_0{}' (\boldsymbol{\mu} - \boldsymbol{\pi}_0 + \mathbf{e}_0) = b_P^t \hat{\rho}^t \boldsymbol{\pi}_0{}' (\boldsymbol{\mu} - \boldsymbol{\pi}_0 + \mathbf{e}_0)$ (264)

where $\Pi_j^{TR\,t} : scalar$ *, trading profit in period 0 of strategy j using estimates from period t*

Proof: Results follow from straightforward application of the definition.

Natural business:

$$\Pi_N^{TR} = \mathbf{Q}_N{}' \mathbf{r}_0 = (b_N \mathbf{X} \mathbf{a}_N)' (\mathbf{X}\boldsymbol{\mu} - \mathbf{X}\boldsymbol{\pi}_0 + \boldsymbol{\upsilon}_0) = b_N \mathbf{a}_N{}' (\boldsymbol{\mu} - \boldsymbol{\pi}_0 + \mathbf{e}_0) \quad \text{(49),(10),(83)} \tag{265}$$

Trading strategies (data and price strategies):

$$\Pi_j^{TR\,t} = \mathbf{Q}_j^{t}{}'\,\mathbf{r}_0 = \left(b_j^t \mathbf{X}\mathbf{a}_j \hat{\boldsymbol{\beta}}_j^t\right)'\mathbf{r}_0 \qquad \text{by (257); (45)} \qquad (266)$$

$$= b_j^t\left(\mathbf{X}\mathbf{a}_j\mathbf{a}_j{}'\left(\boldsymbol{\mu}-\boldsymbol{\pi}_t+\mathbf{e}_t\right)\right)'\left(\mathbf{X}\boldsymbol{\mu}-\mathbf{X}\boldsymbol{\pi}_0+\boldsymbol{\upsilon}_0\right) \qquad \text{by (82), (80)} \qquad (267)$$

$$= b_j^t\left(\boldsymbol{\mu}-\boldsymbol{\pi}_t+\mathbf{e}_t\right)'\mathbf{a}_j\mathbf{a}_j{}'\mathbf{X}'\left(\mathbf{X}\boldsymbol{\mu}-\mathbf{X}\boldsymbol{\pi}_0+\boldsymbol{\upsilon}_0\right) \qquad \text{transposing} \qquad (268)$$

$$= b_j^t\left(\boldsymbol{\mu}-\boldsymbol{\pi}_t+\mathbf{e}_t\right)'\mathbf{a}_j\mathbf{a}_j{}'\left(\boldsymbol{\mu}-\boldsymbol{\pi}_0+\mathbf{e}_0\right) \qquad \text{by (10), (83). Apply price, data} \qquad \#\ (269)$$

5.2 Long term and short term profit

Preamble: We use the technical results derived in section 4.4 to derive profit for each strategy type. We denote expected profit in bold: $\boldsymbol{\Pi} = \mathrm{E}\left[\Pi\right]$.

Result 5.3: Expected long term profit

Data: $$\boldsymbol{\Pi}_X^{LR} = \left(\boldsymbol{\mu}-\bar{\boldsymbol{\pi}}\right)'\mathbf{H}_X\left(\boldsymbol{\mu}-\bar{\boldsymbol{\pi}}\right) \qquad (270)$$

Price: $$\boldsymbol{\Pi}_P^{LR} = \left(\boldsymbol{\mu}-\bar{\boldsymbol{\pi}}\right)'\mathbf{H}_P\left(\boldsymbol{\mu}-\bar{\boldsymbol{\pi}}\right)+\mathbf{q}_{PD}{}'\left(\boldsymbol{\mu}-\bar{\boldsymbol{\pi}}\right) \qquad (271)$$

$$\boldsymbol{\Pi}_P^{LR} = \rho\,\mathbf{u}\left(\boldsymbol{\mu}-\bar{\boldsymbol{\pi}}\right) \qquad (272)$$

N: $$\boldsymbol{\Pi}_N^{LR} = b_N\mathbf{a}_N{}'\left(\boldsymbol{\mu}-\bar{\boldsymbol{\pi}}\right) \qquad \textit{by (262).} \qquad (273)$$

Proof: Use the definition of long term return at (259). Given $\left(\boldsymbol{\mu}-\bar{\boldsymbol{\pi}}\right)$ is constant we write:

$$\boldsymbol{\Pi}^{LR} = \mathrm{E}\left[\textit{quantity vector}\cdot\left(\boldsymbol{\mu}-\bar{\boldsymbol{\pi}}\right)\right] = \mathrm{E}\left[\textit{quantity vector}\right]\left(\boldsymbol{\mu}-\bar{\boldsymbol{\pi}}\right) \qquad (274)$$

so $$\boldsymbol{\Pi}_X^{LR} = \mathrm{E}\left[\mathbf{q}_{X0}\right]'\left(\boldsymbol{\mu}-\bar{\boldsymbol{\pi}}\right) = \left(\boldsymbol{\mu}-\bar{\boldsymbol{\pi}}\right)'\mathbf{H}_X\left(\boldsymbol{\mu}-\bar{\boldsymbol{\pi}}\right) \qquad \text{by (245)} \qquad (275)$$

now $$\boldsymbol{\Pi}_P^{LR} = \mathrm{E}\left[\sum_{t=1}^{\infty} b_P^t\hat{\rho}^t\boldsymbol{\pi}_0{}'\left(\boldsymbol{\mu}-\bar{\boldsymbol{\pi}}\right)\right] \qquad \text{by (264)} \qquad (276)$$

$$= \mathrm{E}\left[\rho_0\boldsymbol{\pi}_0\right]\left(\boldsymbol{\mu}-\bar{\boldsymbol{\pi}}\right) \qquad \text{by (64)} \qquad (277)$$

$$= \rho\,\mathbf{u}\left(\boldsymbol{\mu}-\bar{\boldsymbol{\pi}}\right) \qquad \text{by (233)} \qquad (278)$$

or $$= \left(\left(\boldsymbol{\mu}-\bar{\boldsymbol{\pi}}\right)\mathbf{H}_P+\mathbf{q}_{PD}\right)\left(\boldsymbol{\mu}-\bar{\boldsymbol{\pi}}\right) \qquad \text{by (234)} \qquad \#\,(279)$$

Result 5.4: Data strategy expected long term profit. *Expected long term profit of data strategy j is given by:*

$$\mathbf{\Pi}_j^{LR} = b_j\left(\left(\mathbf{q}_N + \mathbf{q}_{PD}\right)' \boldsymbol{\lambda}^{-1}\mathbf{a}_j\right)^2 \tag{280}$$

Proof: Follows immediately from the above. Use eigenvector coordinate system to diagonalize estimation matrix **H**.

$$\mathbf{\Pi}_j^{LR} = \left(\boldsymbol{\mu} - \bar{\boldsymbol{\pi}}\right)' \mathbf{H}_j \left(\boldsymbol{\mu} - \bar{\boldsymbol{\pi}}\right) \qquad \text{by (270)} \tag{281}$$

$$= \left(\mathbf{q}_N + \mathbf{q}_{PD}\right)' \mathbf{H}^{-1}\mathbf{H}_j\mathbf{H}^{-1}\left(\mathbf{q}_N + \mathbf{q}_{PD}\right) \qquad \text{by (229)} \tag{282}$$

$$= \left(\mathbf{q}_N + \mathbf{q}_{PD}\right)' \boldsymbol{\lambda}^{-1}\left(b_j\mathbf{a}_j\mathbf{a}_j'\right)\boldsymbol{\lambda}^{-1}\left(\mathbf{q}_N + \mathbf{q}_{PD}\right) \qquad \text{by (107)} \qquad \# \tag{283}$$

Remark: We see that the long term profit $\left(\left(\mathbf{q}_N + \mathbf{q}_{PD}\right)' \boldsymbol{\lambda}^{-1}\mathbf{a}_j\right)^2$ varies directly with the correlation between strategy $\mathbf{a}_j$ with the price distortion $\mathbf{q}_N + \mathbf{q}_{PD}$, and inversely with the corresponding weight of money given by eigenvalues $\boldsymbol{\lambda}$. We interpret these factors as the *salience* and *scarcity* of the strategy.

Result 5.5: Natural business expected short term profit. *Agent profit is given by:*

$$\mathbf{\Pi}_N^{SR} = 0 \tag{284}$$

Proof: $\mathbf{\Pi}_N^{SR} = \mathrm{E}\left[b_N\mathbf{a}^{N\prime}\left(\bar{\boldsymbol{\pi}} - \boldsymbol{\pi}_0 + \mathbf{e}_0\right)\right] = b_N\mathbf{a}_N'\mathrm{E}\left[\bar{\boldsymbol{\pi}} - \boldsymbol{\pi}_0 + \mathbf{e}_0\right] = 0 \qquad \text{by (262)} \qquad \# \; (285)$

Result 5.6: Data strategy expected short term profit.

$$\mathbf{\Pi}_j^{SR} = \rho b_j\mathbf{a}_j'\mathbf{H}^{-1}\mathbf{\Sigma}\mathbf{a}_j < 0 \tag{286}$$

Proof: We now apply the results for infinite series derived in Section 4.

$$\Pi_j^{SR} = \sum_{t=1}^{\infty} b_j w(1-w)^{t-1}\left(\boldsymbol{\mu} - \boldsymbol{\pi}_t + \mathbf{e}_t\right)' \cdot \mathbf{a}_j\mathbf{a}_j'\left(\bar{\boldsymbol{\pi}} - \boldsymbol{\pi}_0 + \mathbf{e}_0\right) \qquad \text{by (263), (260)} \tag{287}$$

$$= \sum_{t=1}^{\infty} b_j w(1-w)^{t-1}\left(\boldsymbol{\mu} - \boldsymbol{\pi}_t + \mathbf{e}_t\right)'\mathbf{a}_j\mathbf{a}_j'\left(\bar{\boldsymbol{\pi}} - \boldsymbol{\pi}_0 + \mathbf{e}_0\right) \qquad \text{by (105)} \tag{288}$$

$$= \operatorname{tr}\left(\sum_{t=1}^{\infty} b_j w(1-w)^{t-1} \mathbf{a}_j \mathbf{a}_j{}' \left(\bar{\boldsymbol{\pi}} - \boldsymbol{\pi}_0 + \mathbf{e}_0 \right)\left(\boldsymbol{\mu} - \boldsymbol{\pi}_t + \mathbf{e}_t \right)' \right) \quad \text{trace cyclic property} \qquad (289)$$

so $$\boldsymbol{\Pi}_j^{SR} = \operatorname{tr}\left(b_j \mathbf{a}_j \mathbf{a}_j{}' \cdot \rho \mathbf{H}^{-1} \boldsymbol{\Sigma} \right) \quad \text{by (174). } \boldsymbol{\mu} - \bar{\boldsymbol{\pi}} \text{ component goes out} \qquad (290)$$

$$= b_j \rho \cdot \operatorname{tr}\left(\mathbf{a}_j{}' \mathbf{H}^{-1} \boldsymbol{\Sigma} \mathbf{a}_j \right) \quad \text{factor, use trace cyclic property} \qquad \# (291)$$

Remark: We give an intuitive interpretation of this negative short term profit for data watchers. Analyze the SR profit equation for an investor who reestimated in period 1:

$$\Pi_j^{1\ SR} = \left(\boldsymbol{\mu} - \boldsymbol{\pi}_1 + \mathbf{e}_1 \right)' \mathbf{H}_X \left(\bar{\boldsymbol{\pi}} - \boldsymbol{\pi}_0 + \mathbf{e}_0 \right) \qquad (292)$$

$$= \left(\boldsymbol{\mu} - \boldsymbol{\pi}_1 + \mathbf{e}_1 \right)' \mathbf{H}_X \left(\bar{\boldsymbol{\pi}} + \mathbf{e}_0 - \boldsymbol{\pi}_1 + \frac{w}{\rho} \mathbf{H} \left(\bar{\boldsymbol{\pi}} - \boldsymbol{\pi}_1 + \mathbf{e}_1 \right) \right) \quad \text{sub for } \boldsymbol{\pi}_0 \text{ using (152)} \qquad (293)$$

$$= \left(\boldsymbol{\mu} - \boldsymbol{\pi}_1 + \mathbf{e}_1 \right)' \mathbf{H}_X \left(\bar{\boldsymbol{\pi}} - \boldsymbol{\pi}_1 + \mathbf{e}_0 \right) - \left(\boldsymbol{\mu} - \boldsymbol{\pi}_1 + \mathbf{e}_1 \right)' \mathbf{H}_X \ddot{\boldsymbol{\lambda}} \left(\bar{\boldsymbol{\pi}} - \boldsymbol{\pi}_1 \right) - \left(\boldsymbol{\mu} - \boldsymbol{\pi}_1 + \mathbf{e}_1 \right)' \mathbf{H}_X \ddot{\boldsymbol{\lambda}} \mathbf{e}_1 \qquad (294)$$

using $\ddot{\mathbf{H}}$ at (162), put in diagonalized form $\ddot{\boldsymbol{\lambda}}$ for clarity.

The first term of (294) is what the investor would receive if the price remained at $\boldsymbol{\pi}_1$ and it is positive in line with the investor's expectations. The second term is the movement of price towards the mean in response to the investor's demand following estimation. It is negative but not so large as to eliminate the expected profit. The third term isolates the effect of the error term $\mathbf{e}_1$, which causes the investors' estimates to be inaccurate and a price move in the direction of the inaccuracy. Because the error term is relatively large, this term is strongly negative for the investor.

Result 5.7: Price strategy expected short term profit.

$$\boldsymbol{\Pi}_P^{SR} = -\rho \operatorname{tr}\left(\mathbf{H}_X \mathbf{H}^{-1} \boldsymbol{\Sigma} \right) > 0 \quad \text{by (261) this is opposite to (286).} \qquad (295)$$

Short term profit simplified version: The short term profit expression can be simplified. We simplify the expression for the price variance matrix $\boldsymbol{\Sigma}$ as follows:

$$\boldsymbol{\Sigma} = \sigma^2 \left(-\frac{w}{\rho} \mathbf{H} \right) \left(2\mathbf{I} + \frac{w}{\rho} \mathbf{H} \right)^{-1} \approx \sigma^2 \left(-\frac{w}{\rho} \mathbf{H} \right) \cdot \frac{1}{2} \cdot \left(\mathbf{I} - \frac{w}{2\rho} \mathbf{H} \right) \quad \text{by (160), (135)} \qquad (296)$$

$$\approx -\frac{w\sigma^2\mathbf{H}}{2\rho} \qquad \text{casting out the higher order term in } w \qquad (297)$$

so $$\mathbf{\Pi}_j^{SR} = \rho b_j \mathbf{a}_j{}'\mathbf{H}^{-1}\left(-\frac{w\sigma^2\mathbf{H}}{2\rho}\right)\mathbf{a}_j \qquad \text{substituting into (286) using (297)} \qquad (298)$$

$$= -\tfrac{1}{2} b_j w\sigma^2 \qquad \text{simplifying and using (14)} \qquad (299)$$

so $$\mathbf{\Pi}_X^{SR} = \sum_{j \in X}\left(-\tfrac{1}{2} b_j w\sigma^2\right) = -\tfrac{1}{2} b_X w\sigma^2 \qquad \text{by (36)} \qquad (300)$$

$$\mathbf{\Pi}_P^{SR} = -\mathbf{\Pi}_X^{SR} = \tfrac{1}{2} b_X w\sigma^2 \qquad \text{by (295)} \qquad (301)$$

These expressions have the benefit of being expressible purely in terms of parameters. The omission of the second term leads to a slight underestimate of typical values of 5-20%.

Simulation: Simulation shows a close fit between the theoretical expressions for profit given above and simulation values. Results available on request from the author.

5.3 Costs and economic equilibrium

Definitions: Operational cost. In addition to trading results there is another component of profit: operational costs, which includes a notional charge for normal profit. We define costs for strategy *j* per estimation period as:

$$E_j^{OP} = (Total\,Costs + Normal\ Profit)_j < 0 : scalar \text{, total costs for } j \qquad (302)$$

$$C_j^{OP} = \frac{E_j^{OP}}{T_j} < 0 : scalar \text{, costs per unit weight of money for strategy } j \qquad (303)$$

Using (35), total costs for strategy *j* in relative terms (per unit total trade T_{TR}) are given by:

$$\frac{E^{OP}}{T_{TR}} = \left(\frac{T_j}{T_{TR}}\right)\cdot\left(\frac{E^{OP}}{T_j}\right) = b_j C_j^{OP} < 0 : scalar \text{, relative costs for strategy } j \qquad (304)$$

Premise 11: cost structure. There are no fixed costs. Costs per unit take a constant negative value for every trading strategy including the price strategy.

$$C_j^{OP} = C_P^{OP} = C^{OP} < 0 : scalar \qquad (305)$$

where $C^{OP} = \sum_{j \in X,P} C_j^{OP} b_j : scalar$: total operational cost across trading strategies (306)

Definitions: Net costs and root costs. Define net costs and root costs for trading strategies:

$$C_j = C_j^{OP} - \tfrac{1}{2} w\sigma^2 < 0 : scalar$$, costs per unit b_j including short term loss (307)

$$C_P = C_P^{OP} - \tfrac{1}{2} w\sigma^2 < 0 : scalar$$, costs per b_P including short term component (308)

$$c_j = -\sqrt{-C_j} < 0 : scalar$$, root cost: square root of cost defined at (307) (309)

$$c_j = c_X < 0 \qquad \text{for all } j \in X, \text{ by (305)} \qquad (310)$$

Premise 12: Equilibrium. There is perfect competition in the information market and no supernormal profits. At economic equilibrium, traders break even after accounting for normal profit as a charge to costs. A trading strategy with positive weight of money $b_j > 0$ at equilibrium is referred to as *viable*.

$$\mathbf{\Pi}_j = \mathbf{\Pi}_j^{LR} + \mathbf{\Pi}_j^{SR} + C_j^{OP} b_j = 0 \qquad (311)$$

In the case of natural business there are no operational or short term profit costs, but as shown there is an expected trading loss from accepting the price. We evaluate the natural business cost criterion set out in (47):

$$\mathbf{\Pi}_N = b_N C_N \qquad \text{by (53)} \qquad (312)$$

where $C_N = \mathbf{a}_N{}' (\boldsymbol{\mu} - \bar{\boldsymbol{\pi}}) : scalar$, expected cost per unit by (273) (313)

Result 5.8: Market economy theorem. *At equilibrium, the natural business trading loss exactly equals trading costs.*

$$C^{OP} = C_N b_N \qquad (314)$$

Proof: We state the profits for each strategy type and see how the elements are related:

$$(\boldsymbol{\mu} - \bar{\boldsymbol{\pi}})' \mathbf{H}_j (\boldsymbol{\mu} - \bar{\boldsymbol{\pi}}) + \mathrm{E}\left[\Pi_j^{SR}\right] = -C_j^{OP} b_j \qquad (270), (311) \qquad (315)$$

$$(\boldsymbol{\mu} - \bar{\boldsymbol{\pi}})' \mathbf{H}_P (\boldsymbol{\mu} - \bar{\boldsymbol{\pi}}) + \mathbf{q}^{PD\prime} (\boldsymbol{\mu} - \bar{\boldsymbol{\pi}}) + \mathrm{E}\left[\Pi_P^{SR}\right] = -C_P^{OP} b_P \qquad (271), (311) \qquad (316)$$

$$b_N \mathbf{a}_N' \left(\boldsymbol{\mu} - \bar{\boldsymbol{\pi}}\right) = C_N b_N \qquad \text{(312), (313)} \qquad (317)$$

Add the equations for every trading agent *j* and natural business agent *k*;

now $\left(\boldsymbol{\mu} - \bar{\boldsymbol{\pi}}\right)' \mathbf{H}\left(\boldsymbol{\mu} - \bar{\boldsymbol{\pi}}\right) + \left(\mathbf{q}_N + \mathbf{q}_{PD}\right)' \left(\boldsymbol{\mu} - \bar{\boldsymbol{\pi}}\right) = 0$ **(229)**$\times\left(\boldsymbol{\mu} - \bar{\boldsymbol{\pi}}\right)$ (318)

and $\boldsymbol{\Pi}_X^{SR} + \boldsymbol{\Pi}_P^{SR} = 0$ by (261). So LHS of summed equations is 0. (319)

so $-\sum_{j \in X,P} C_j^{OP} b_j + C_N b_N = 0$ summing RHS of equations. Apply (306). # (320)

Corollary 5.9: *A market must include natural business to be viable*.

Proof: Given (305), LHS (314) is negative. Equality requires $b_N > 0$. # (321)

5.4 Strategy-return correlation

We show that the strategy-return correlation at equilibrium is negative.

Definition: strategy-return correlation. The strategy-return correlation is given by:

$$r_j = \frac{\mathbf{a}_j' \left(\boldsymbol{\mu} - \bar{\boldsymbol{\pi}}\right)}{\left|\boldsymbol{\mu} - \bar{\boldsymbol{\pi}}\right|} : scalar$$, note that $\mathbf{a}_j$ is normalized, $\boldsymbol{\mu} - \bar{\boldsymbol{\pi}}$ is not. (322)

Result 5.10: Negative strategy-return correlation. *The strategy-return correlation for a data strategy is given by, in the coordinate system defined by eigenvectors* $\boldsymbol{\Lambda}$ *:*

$$\mathbf{a}_j' \left(\boldsymbol{\mu} - p\,\bar{\boldsymbol{\pi}}\right) = -b_N \mathbf{a}_j' \boldsymbol{\lambda}^{-1} \mathbf{a}_N \qquad (323)$$

where $p = \left(\mathbf{I} + \dfrac{b_P \sigma_C^2 \mathbf{H}^{-1}}{\left|\bar{\boldsymbol{\pi}}\right|}\right) \approx \mathbf{I} : N \times N$ (324)

Proof: Apply the Expected price equation to $\left(\boldsymbol{\mu} - \bar{\boldsymbol{\pi}}\right)$ and simplify:

$\mathbf{a}_j' \left(\boldsymbol{\mu} - \bar{\boldsymbol{\pi}}\right) = \mathbf{a}_j' \left(-b_N \mathbf{H}^{-1} \mathbf{a}_N + b_P \sigma_C^2 \mathbf{H}^{-1} \mathbf{u}\right)$ by (230). Rearrange terms to get: (325)

$\mathbf{a}_j' \left(\boldsymbol{\mu} - \left(\mathbf{I} + \dfrac{b_P \sigma_C^2 \mathbf{H}^{-1}}{\left|\bar{\boldsymbol{\pi}}\right|}\right) \bar{\boldsymbol{\pi}}\right) = -b_N \mathbf{a}_j' \mathbf{H}^{-1} \mathbf{a}_N$ where $\mathbf{a}_j$ is a data strategy (326)

$\mathbf{a}_j' \left(\boldsymbol{\mu} - p\,\bar{\boldsymbol{\pi}}\right) = -b_N \mathbf{a}_j' \boldsymbol{\lambda}^{-1} \mathbf{a}_N$, variables $\mathbf{a}_j, \mathbf{a}_N$ transformed by eigenvectors $\boldsymbol{\Lambda}$ # (327)

Given $\boldsymbol{\lambda}^{-1} > 0$ by (134) and data strategies are positively correlated to natural business by

(54), it is likely, although not certain, that $\mathbf{a}_j(\boldsymbol{\mu} - p\bar{\boldsymbol{\pi}}) < 0$ and consequently the strategy-return correlation $\mathbf{a}_j(\boldsymbol{\mu} - \bar{\boldsymbol{\pi}})$ is negative. In simulation, the correlation between strategy and return is invariably negative after the initial stage. We assume that:

$$\mathbf{a}_j'(\boldsymbol{\mu} - \bar{\boldsymbol{\pi}}) \le 0, \quad j \in X \tag{328}$$

5.5 Economic equilibrium

A theory of market structure should explain the market share of the viable strategies at equilibrium. To this end we seek expressions for the market shares b_j of each strategy, given so far as possible in terms of the structural variables which define the problem. Because of the non-linear nature of the problem, completely structural expressions are not possible (or at least not achieved here). The results presented below incorporate the price coefficient ρ, but they do not require the market price variable $\bar{\boldsymbol{\pi}}$.

Preamble: The following result rearranges data profit to show that at equilibrium, expected price $\bar{\boldsymbol{\pi}}$ is related in a very simple way to the parameters of the system.

Result 5.11: Equilibrium price relationships. *The following relationships between costs and market variables hold at economic equilibrium:*

data: $\mathbf{a}_j'(\boldsymbol{\mu} - \bar{\boldsymbol{\pi}}) = c_j$ *where* $j \in X$ (329)

price: $\rho\mathbf{u}'(\boldsymbol{\mu} - \bar{\boldsymbol{\pi}}) + b_P C_P + \tfrac{1}{2} w\sigma^2 = 0$ (330)

N: $b_N \mathbf{a}_N'(\boldsymbol{\mu} - \bar{\boldsymbol{\pi}}) = C^{OP}$ *by (317); (314)* (331)

Proof: Sum the components of profit, and apply equilibrium criterion (311).

Data: $\boldsymbol{\Pi}_j = (\boldsymbol{\mu} - \bar{\boldsymbol{\pi}})'\left(b_j \mathbf{a}_j \cdot \mathbf{a}_j'\right)(\boldsymbol{\mu} - \bar{\boldsymbol{\pi}}) + b_j C_j^{OP} - \tfrac{1}{2} b_j w\sigma^2$ by (270), (303), (299) (332)

so $b_j\left(\mathbf{a}_j'(\boldsymbol{\mu} - \bar{\boldsymbol{\pi}})\right)^2 + b_j C_j = 0$ by (307) and equilibrium condition (311) (333)

$\left(\mathbf{a}_j'(\boldsymbol{\mu} - \bar{\boldsymbol{\pi}})\right)^2 = -C_j$ simplifying (334)

$$\mathbf{a}_j'\left(\boldsymbol{\mu}-\overline{\boldsymbol{\pi}}\right)=c_j$$ by (309), (328) (335)

Price: $$\boldsymbol{\Pi}_P=\rho\,\mathbf{u}\left(\boldsymbol{\mu}-\overline{\boldsymbol{\pi}}\right)+\tfrac{1}{2}\left(1-b_p\right)w\sigma^2+b_PC_P^{OP}$$ by (272), (303), (301), (39) (336)

$$=\rho\,\mathbf{u}\left(\boldsymbol{\mu}-\overline{\boldsymbol{\pi}}\right)+\tfrac{1}{2}w\sigma^2+b_P\left(C_P^{OP}-\tfrac{1}{2}w\sigma^2\right)$$ rearranging. Use (308) for result. # (337)

Definitions: Strategy matrices. Define the following matrices for the data strategies:

$$\mathbf{b}_X=\begin{bmatrix}b_1\\ b_2\end{bmatrix}: N\times 1$$, *vector of data strategy weights of money* (338)

$$\mathbf{A}_X=\begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_N\end{bmatrix}: N\times N$$, *matrix of N independent data strategy vectors* (339)

$$\mathbf{C}_X=\begin{bmatrix}c_1 & \cdot\\ \cdot & c_N\end{bmatrix}: N\times N$$, *diagonal matrix of data strategy root costs* (340)

$$\mathbf{C}_{VEC}=\begin{bmatrix}c_1\\ c_N\end{bmatrix}: N\times 1$$, *vector of data strategy root costs* (341)

$$\boldsymbol{\beta}^{RET}=\left(\mathbf{A}_X'\mathbf{A}_X\right)^{-1}\mathbf{A}_X'\left(b_N\mathbf{a}_N+\rho\,\boldsymbol{\mu}\right): N\times 1$$, regression coefficient (342)

for the regression of $N\times 1$ vector $b_N\mathbf{a}_N+\rho\,\boldsymbol{\mu}$ onto the data strategy vectors.

$$\boldsymbol{\Omega}=\mathbf{A}_X'\mathbf{A}_X : N\times N$$, variance-covariance matrix of the data strategies. (343)

Result 5.12: Equilibrium market share theorem. *Given premises 1-13, the weight of money of each strategy at economic equilibrium is given by:*

$$\mathbf{b}_X\geq\mathbf{0}_{N\times 1}$$ (344)

$$\overline{\boldsymbol{\pi}}'=\boldsymbol{\mu}'-\mathbf{C}_{VEC}'\mathbf{A}_X^{-1}$$ (345)

X: $$\mathbf{b}_X=-\mathbf{C}_X^{-1}\boldsymbol{\beta}^{RET}+\rho\,\mathbf{C}_X^{-1}\boldsymbol{\Omega}^{-1}\mathbf{C}_{VEC}$$ (346)

P: $$\rho\,\mathbf{C}_{VEC}'\mathbf{A}_X^{-1}\mathbf{u}+b_PC_P+\tfrac{1}{2}w\sigma^2=0$$ (347)

N: $$b_N\mathbf{C}_{VEC}'\mathbf{A}_X^{-1}\mathbf{a}_N=C^{OP}$$ (348)

In the special case where $\boldsymbol{\Omega}=\mathbf{I}_{N\times N}$ *and* $\mathbf{C}_X=c_j\mathbf{I}_{N\times N}$. *equation (346) can also be written:*

$$b_j = \frac{\boldsymbol{\beta}_j^{RET}}{-c_j} + \rho \qquad (349)$$

Proof: Apply the previous strategy equilibrium results to the expected price equation (235).

$$\mathbf{A}_X{}'\left(\boldsymbol{\mu} - \bar{\boldsymbol{\pi}}\right) = \mathbf{C}_{VEC}$$ assemble (329) for N data strategies (350)

$$\bar{\boldsymbol{\pi}} = \boldsymbol{\mu} - \left(\mathbf{A}_X{}'\right)^{-1} \mathbf{C}_{VEC}$$ rearrange and invert (351)

Use these results to derive the expressions:

N: $b_N \left(\boldsymbol{\mu} - \bar{\boldsymbol{\pi}}\right)' \mathbf{a}_N = C^{OP}$ by (331). Substitute in (351) for result. (352)

Price: $\rho\left(\boldsymbol{\mu} - \bar{\boldsymbol{\pi}}\right)' \mathbf{u} + b_P C_P + \tfrac{1}{2} w\sigma^2 = 0$ by (330). Substitute in (351) for result. (353)

Data: $\sum_{j \in X} b_j \mathbf{a}_j \mathbf{a}_j{}' \left(\boldsymbol{\mu} - \bar{\boldsymbol{\pi}}\right) + \rho \frac{\bar{\boldsymbol{\pi}}}{|\bar{\boldsymbol{\pi}}|} + b_N \mathbf{a}_N = \mathbf{0}_{N\times 1}$ by (236) and $\mathbf{u} = \frac{\bar{\boldsymbol{\pi}}}{|\bar{\boldsymbol{\pi}}|}$ (354)

Near economic equilibrium we can take it that $|\bar{\boldsymbol{\pi}}| \doteq 1$. Accordingly, (355)

$$\sum_{j \in X} b_j \mathbf{a}_j c_j + \rho\left(\boldsymbol{\mu} - \left(\mathbf{A}_X{}'\right)^{-1} \mathbf{C}_{VEC}\right) + b_N \mathbf{a}_N = \mathbf{0}_{N\times 1}$$ by (329), (351) (356)

$$\mathbf{A}_X \mathbf{C}_X \mathbf{b}_X + \rho\boldsymbol{\mu} - \rho\left(\mathbf{A}_X{}'\right)^{-1} \mathbf{C}_{VEC} + b_N \mathbf{a}_N = \mathbf{0}_{N\times 1}$$ rearrange LHS term (357)

$$\mathbf{C}_X \mathbf{b}_X + \left(\mathbf{A}_X{}'\mathbf{A}_X\right)^{-1} \mathbf{A}_X{}'\left(\rho\boldsymbol{\mu} + b_N \mathbf{a}_N\right) - \rho\left(\mathbf{A}_X{}'\mathbf{A}_X\right)^{-1} \mathbf{C}_{VEC} = \mathbf{0}_{N\times 1} \qquad (358)$$

multiplying by $\left(\mathbf{A}_X{}'\mathbf{A}_X\right)^{-1} \mathbf{A}_X{}'$. Strategies in $\mathbf{A}_N$ independent by (339).

$$\mathbf{C}_X \mathbf{b}_X + \boldsymbol{\beta}^{RET} - \rho\boldsymbol{\Omega}^{-1}\mathbf{C}_{VEC} = \mathbf{0}_{N\times 1}$$ by (342), (343) # (359)

Remark: To solve this system we must obtain price coefficient ρ from another source. Here we use the value generated by the simulation; a parametric method is also available.

5.6 Simulation: market share

The market structure theory developed above is tested by making predictions of the weight of money b_j, the expected price $\bar{\boldsymbol{\pi}}$ and the price coefficient ρ at equilibrium.

Procedure: We use a market evolution procedure whereby the simulation is broken into consecutive runs of 100,000 estimation periods. In each run, the market shares of the strategies are altered according to their profitability in the preceding run. In this way the system converges to zero profits for each trading strategy. Some data strategies prove to be non-viable, i.e. their market share drops to zero.

The simulation has three phases. In the first phase of each simulation, strategies where the weight of money b_j falls to less than 0.0010 are replaced by new data strategies chosen at random. Over time the positively correlated strategies are eliminated for the reasons given at (328). In the second phase of the simulation, the set of strategies are held constant. The profit of each viable strategy gravitates to zero and their market share stabilizes. The weight of money of unviable strategies falls to zero. Finally a run of 10,000,000 iterations is made to provide estimates with minimal variance.

Parameters: Basic parameters are as follows. Data strategies $\{\mathbf{a}_j\}$ chosen randomly.

$$N = 12 \quad J = 17 \quad \boldsymbol{\mu} = \begin{bmatrix} 1 \\ . \\ . \end{bmatrix} \quad \mathbf{a}_N = \begin{bmatrix} 0.6 \\ 0.8 \\ . \end{bmatrix} \quad w = 0.002 \quad \sigma = 0.01$$
$$C_j = -0.000050, \quad j \in X, P \tag{360}$$

Table 5.3: Comparison of theoretical and simulation equilibrium price. The table compares the theoretical price at equilibrium (345) with the simulation price at equilibrium, and also the theoretical price for the given market shares found using (231).

The row show results for each dimension of coefficient space. The final row shows the magnitude of each vector calculated using the Euclidean norm.

The columns present vector information as follows:

pi the PD Theoretical expected price (231), price distortion component.

pi the N Theoretical expected price (231), natural business component.

pi the Theoretical expected price (231), sum of the two previous components and $\boldsymbol{\mu}$.

pi sim Simulation value of the expected price, average of the iteration values.

err The difference between the theoretical expected price and the simulated value.

err % The difference expressed as a percentage of the simulated value, for dim 1 only.

pi prPara Not relevant here.

err Not relevant here.

pi prSim Simulation-augmented prediction of expected price at equilibrium, using (345).

err The difference between the prediction and the simulated value.

The table shows that both theoretical price concepts are close to the measured value.

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Table 5.4: Comparison of theoretical and simulated equilibrium market shares. Two tables are given. The first table shows the actual and predicted market shares after completion of the first phase described above. All strategies have negative strategy-return correlations, but the market is not in economic equilibrium. The second table shows the situation after the market has moved to economic equilibrium.

The columns present profit and other information for each strategy as follows:

corr Correlation between the strategy vector $\mathbf{a}_N, \mathbf{a}_j, \mathbf{u}$ and expected return $\boldsymbol{\mu} - \bar{\boldsymbol{\pi}}$.

pred: para Predictions obtained parametrically using methods not described in this paper.

pred: sim Prediction of market share, using simulation value of ρ is used and (346) - (348)

b Current market share b_j

b diff The amount by which market share b_j has changed from the previous run.

Simulation profit values:

pi sr Simulation value for expected short run profit, average over iterations.

pi lr Simulation value for expected long run profit.

pi trade Simulation value for expected trading profit, sum of SR run and LR values.

pi cost The cost component of profit. On a per unit basis, cost is $C_j = -0.000050$

pi net Simulation value for expected net profit, sum of trading profit and costs.

Profit amounts in the table are multiplied by 1000 for clarity. The values for natural business, trading total and grand total are absolute amounts. The values for the trading strategies are per unit amounts, i.e. $\mathbf{\Pi}_j / b_j$. Where $b_j < 0.00005$ a dash is shown.

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Table 5.4.1: End of phase 1. All strategies have negative strategy-return correlation.

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Table 5.4.2: End of phase 2. Market in economic equilibrium. 'Pi net' values close to 0.

Discussion of results: We make the following observations:

(i) In Table 5.3.1 End of phase 1, the predictions given in 'pred: sim' column, are quite distinct from the simulation values of b_j given in 'b' column. The market is not in equilibrium, i.e. strategy profits given in the 'pi net' column are not close to zero.

(ii) In Table 5.3.2 End of phase 2, the market is in equilibrium. The predictions have changed from the Phase 1 values, and deviate from realized results by about 3% for the data strategies and 10% for the price strategy. The sum of all strategies shares is close to 1.0000 as per (38).

Summary: Simulation results are consistent with the market structure theory developed above. On this basis we regard the market structure theory as validated.

6. Conclusion

6.1 The nature of security market processes: heterogeneous least squares learning

Typically economic studies of security market equilibrium view security valuation as a black box process. Valuation is carried out by a subset of traders known as fundamental analysts or inside traders, and it is neither possible nor necessary to investigate the process further. An alternative paradigm is offered here. Information is viewed here as something like a field as that term is understood in physics, in that it permeates the market and governs the behaviour of the agents. All traders including price traders analyse market information, and all incur cost in doing so. An agent does not 'add together' information to arrive at a value for the security in the way that, for instance, an engineer building a bridge adds together information from first principles. Rather, an agent looks at security value and 'subtracts out' relevant factors using regression, leaving a residual component of value (the residual term in the regression) which remains unexplained.

From this premise it has been shown:

(i) The market will generate a regression coefficient for the price variable which is negative. This allows the market to 'make a price', i.e. there is a price which balances supply and demand.

(ii) Notwithstanding that no agent possesses all the information and each agent is operating in isolation, a equilibrium exists in the security market whereby the price is efficient. This occurs regardless of how many agents are looking at each type of information. At *financial* equilibrium, the weight of money behind each source of information is *not* necessarily proportional to that information's salience.

(iii) An equilibrium exists in the information market whereby each agent makes a normal profit. The market is sustainable. At *economic* equilibrium, the weight of money behind each source of information *is* proportional to that information's salience.

6.2 The coefficient model

From the regression market assumption we proceed to define coefficient space – a mathematical space from which the investor data set $\mathbf{X}$ has been eliminated, and the market parameters of return, price, quantity and profit can be depicted as vectors and inner products. Coefficient space clarifies the relationships between the variables and processes of a financial market.

There are two key results in the coefficient model. The first is the price equation, (69), which is a version of the demand equation (56) which gives the location of the price vector $\boldsymbol{\pi}$ in coefficient space. The second is the price change principle (109), which shows how the price vector changes in each estimation period. We demonstrate that the market price $\boldsymbol{\pi}$ converges to the efficient point (143) using the price change principle.

This principle underlying efficiency, the price change principle, is quite different from the principle underlying convergence in macroeconomic learning models, where agents adjust their expectations in the direction of realized values. Here each agent is estimating different components of the return parameter $\boldsymbol{\mu}$ and they never observe their component of return $\boldsymbol{\mu}$ in isolation, or resolve the estimation bias caused by the correlation of their variables with other components of $\boldsymbol{\mu}$. The price system mediates to bring demand into balance. Because return predictions based on old estimates sum to zero, price is pushed in the direction of the new and more accurate estimates. The market operates as a blind multi-pronged hill climbing algorithm which converges to security value. However, the efficient market is a journey and not a destination; statistical error results in the price trajectory rotating around value rather than coinciding with it.

The expression for the price coefficient reveals that it will be negative providing that natural business strategy is positively correlated with return: stability condition (256). Such a positive correlation is in line with apriori expectations. The price coefficient formula has tested by simulation and its validity is demonstrated over a wide range of parameter values.

6.3 The economics of the information market

We have evaluated the profits of each type of market strategy. Profit can be divided into short run (286), (295), (284) and long run (270)-(273) components. Short run profits are negative for data analysts, because of the tendency for the data traders to turn the price against themselves as they act on fresh estimates (294). Nonetheless the long term component of data strategy profit is sufficient to recompense data traders for this short term cost and cover their expenses. We show at (280) that the long term return of a strategy is proportional to the salience of the strategy, as measured by the correlation with return, and inversely proportional to its scarcity, as measured by the strategy's average eigenvalue in the estimation matrix. For price traders the situation is reversed. The short term loss incurred by data traders is received by these traders as a profit and is indeed the largest component of price profit. Both data and price traders incur costs which we have assumed here to be constant variable costs.

The natural business trader takes the market price without analyzing its value in the particular circumstances, and makes a loss on trading. This loss is compensated by the profit which the natural business traders make as a return to holding risky capital. The return to capital does not enter directly into the model, rather we assume that there is a maximum trading loss which the natural business will accept.

At equilibrium, the traders make zero profits net of costs. In this situation the natural business trading loss provides the trading profit of the traders, which offsets the operational costs of the traders.

Market share at equilibrium: By (346), the market share of each data strategy at equilibrium is given by regressing vector $b_N\mathbf{a}_N + \rho\boldsymbol{\mu}$ onto a multiple regression of strategy vectors, i.e. $\boldsymbol{\beta}^{RET} = \left(\mathbf{A}_X{}'\mathbf{A}_X\right)^{-1}\mathbf{A}_X{}'\left(b_N\mathbf{a}_N + \rho\boldsymbol{\mu}\right)$. The vector $b_N\mathbf{a}_N + \rho\boldsymbol{\mu}$ can be interpreted as measuring the distortion introduced by the natural business and price traders. Again we see that market share of a strategy is proportional to its salience and inversely proportional to its scarcity, but these attributes manifest differently. Salience is measured by the degree to which the strategy vector explains the distortion vector. Scarcity is measured by the correlation between the strategy vector and other strategy vectors in coefficient space. Strategy vectors which are strongly correlated with each other share the explanation of the distortion vector between them, and the regression coefficients are thereby attenuated. By contrast, the coefficient of a strategy vector with low correlations to other strategy vectors receives the full impact of the regressand on the strategy vector, and the market share of the strategy is corresponding large. These results are exactly what we expect *a priori* from the disciplinary matrix of economics.

The economics of a security market has been the source of debate. The Grossman-Stiglitz (1980) paradox suggests that an efficient market will not pay for analysis. Here we resolve the paradox by including natural business. The price does not move exactly to return, and the residual discrepancy provides the return to the traders. Although every agent receives the same price, different classes of agent are on different sides of the trading, and natural business tends to pay prices which are slightly higher than perfectly efficient prices. Result (314) shows that at economic equilibrium, the losses made by natural business exactly equal the operational costs and normal profit C^{OP} of the traders. The market gets as much analysis, and consequently efficiency, as natural business is prepared to pay for.

6.4 Testable hypotheses and empirical support from the literature

In a study which investigated the efficient markets hypothesis, Schwert (2002) stated that

"…the size [excess return] effect, the value effect, the weekend effect, and the dividend yield effect seem to have weakened or disappeared after the papers that highlighted them were published." (Swert 2002, abstract)

Schwert's finding is what we would expect. Market prices do not reflect information which is considered by nobody. Once a relevant variable is identified, it is incorporated into investor models and the market becomes efficient with respect to that variable.

6.5 Conclusion: the investor's life in the bush of ghosts

We rejected the assumption that any agents in a financial market receive price signals via exogeneous black-box processes, and assume instead that all agents use least squares learning – a 'regression market'. We have shown that the price in a regression market reflects the sum total of everybody's knowledge, and that each piece of knowledge will be valued correctly regardless of the weight of money behind it. Investor strategies support each other through a network of interlocking expectations. Each investor relies on the pattern of investment of the other investors for the accuracy of their own estimations, and no part can be removed without disrupting the other parts. At equilibrium the average market price $\overline{\boldsymbol{\pi}}$ is close to the return coefficient vector $\boldsymbol{\mu}$. Recall that $\boldsymbol{\mu}$ is defined by:

$$\mathbf{y} = \mathbf{X}\boldsymbol{\mu} + \mathbf{u} \qquad \text{(15) restated} \qquad (361)$$

In effect, the regression market computes a multiple regression to estimate return $\boldsymbol{\mu}$, even though each constituent agent is carrying out regression with only one variable. The ability of a regression market and, we contend, real-world security markets to compute a multiple regression in this fashion can be considered to be a fundamental organising principle of nature.

A theme of economics going back to Adam Smith's invisible hand is that the whole is greater than the sum of the parts. The coefficient model identifies the parts which have previously been obscured. Behind the outward appearance of a market – the trades and current market price - is a hidden substrate where price is not a single piece of information but an object with its own independent existence as an economic entity. We can replace the vague statement that the invisible hand guides the market to its efficient point, and its equally vague modern equivalent, that efficiency is an emergent property of markets, with a well-defined statement. Market efficiency is a property of the market price object rather than a property inherited by the market from informed traders.

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